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Compactification of Moduli Spaces and Mirror Symmetry

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Compactification of Moduli Spaces and Mirror Symmetry

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Compactification of Moduli Spaces and Mirror Symmetry

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Abstract: Olsson gives modular compactifications of the moduli of toric pairs and the moduli of polarized abelian varieties $\mathscr{A}_{g,\delta}$ in [Ols08]. We give alternative constructions of these compactifications by using mirror symmetry. Our constructions are toroidal compactifications. The data needed for a toroidal compactification is a collection of fans. We obtain the collection of fans from the Mori fans of the minimal models of the mirror families. Moreover, we reinterpretate the compactification of $\mathscr{A}_{g,\delta}$ in terms of KSBA stable pairs. We find that there is a canonical set of divisors $S(K_2)$ associated with each cusp. Near the cusp, a polarized semiabelic scheme $(\mathcal{X}, G, \mathcal{L})$ is the canonical degeneration given by the compactification if and only if $(\mathcal{X}, G, \Theta)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$ for any $\Theta \in S(K_2)$. The two compactifications presented here are a part of a general program of applying mirror symmetry to the compactification problem of the moduli of Calabi–Yau manifolds. This thesis contains the results in [Zhu14b] and [Zhu14a].

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Chapter 1

Introduction

1.1 The Basic Questions

We concern two related questions. The first is about the degeneration of Calabi–Yau manifolds:

The Basic Question 1.1. *Given a family of polarized Calabi–Yau manifolds* $(\mathcal{X}^*, \mathcal{L}^*)$ *over a punctured disk* Δ^**, is there a canonical way to fill in the central fiber? If the answer is "yes", then how to produce the canonical central fiber?*

Notice that Question 1.1 is already interesting even for dimension 1.

Example 1.2. *Consider the Hesse family near* 0, $\mathcal{X}^* = \{(x, y, z) \in \mathbf{CP}^2; t(x^3 + y^3 + z^3) + xyz = 0\}$ *over* $0 < |t| < 1$*, with the relatively ample invertible sheaf* $\mathcal{L}^* = \mathcal{O}_{\mathcal{X}^*}(1)$*. Take* $t = 0$ *in the above embedding in* $\mathbf{CP}^2$*, we get the central fiber* $I_3 = \{xyz = 0\}$*, with the line bundle* $\mathcal{L} = \mathcal{O}_{I_3}(1)$*. Notice that the three irreducible components* L_1*,* L_2*,* L_3 *of* I_3 *are all* -2*-curves. By twisting* $\mathcal{L}$ *by* $\mathcal{L} \otimes \mathcal{O}(L_1 + L_2)$*, and contracting* $L_1 + L_2$*, we get* I_1*. Moreover, blow up the* A_2 *singularity* p *of* I_1 *to get a nonreduced exceptional divisor* E[1]*. Then*

[1]This is not the ordinary blow up of the surface along p. Look at the cone σ for the toric

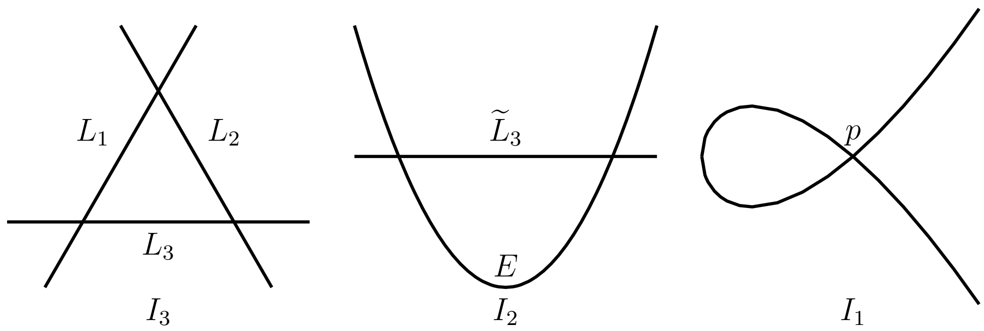

Figure 1.1: The Possible Central Fibers For the Hasse Family.

make a base change of degree 2. *We can extend* $\mathcal{L}$ *such that* $\deg \mathcal{L}|_E = 2$ *and* $\deg \mathcal{L}|_{\widetilde{L}_3} = 1$. *Therefore, the three central fibers in Figure 1.1 are equally good. There is no obvious canonical limit.*

The second one is how to compactify the moduli space $\mathcal{M}$ of Calabi–Yau manifolds. Since a family of Calabi–Yau manifolds may degenerate as shown in Example 1.2, the moduli space $\mathcal{M}$ is not compact. We want to embed $\mathcal{M}$ into a compact space $\overline{\mathcal{M}}$ as a dense open subspace, because there are more techniques available to study the global geometry of compact spaces. Notice that given a non-compact space $\mathcal{M}$ of higher dimensions, the compactification, if exists, is not unique, as we can always do some birational transforms to the boundary. Therefore we wonder

The Basic Question 1.3. *Is there a canonical compactification* $\overline{\mathcal{M}}$ *for the moduli space* $\mathcal{M}$ *of Calabi–Yau manifolds?*

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picture of an A_2 singularity. We add the ray $v + v'$, where v and v' are rays of σ.

Question 1.3 implies Question 1.1 if we require $\overline{\mathcal{M}}$ to be proper and to admit a family extending the universal family over $\mathcal{M}$. The model we keep in mind is the celebrated Deligne–Mumford's compactification of moduli space of curves with marked points $\mathcal{M}_{g,d}$[2] by the moduli space of stable curves with marked points $\overline{\mathcal{M}}_{g,d}$. The problem is, however, we have too many choices for both Question 1.3 and Question 1.1. For Question 1.1, we have too many choices of degenerations if the family does not come with an ample divisor. For Question 1.3, there are too many birational models. Our basic philosophy is that mirror symmetry provides canonical choices for both cases. See Sect. 1.3. In this thesis, we show that this philosophy works well for the compactification problems of the moduli of polarized abelian varieties and the moduli of toric pairs. Even when these two cases have been worked out in [Ols08], we obtain new results.

1.2 The Main Results

In [Ols08], Olsson give positive answers to Question 1.3 for the moduli space of polarized abelian varieties $\mathscr{A}_{g,d}$ and the moduli space of toric pairs. We focus on $\mathscr{A}_{g,d}$ first. The compactification $\overline{\mathscr{A}_{g,d}}$ in [Ols08] uses the AN construction, which is a general construction of the degeneration of abelian varieties over a complete normal domain[3]. The AN construction is complicated, and

[2]Here $g \geqslant 1, d \geqslant 0$ or $g = 1, d \geqslant 1$.

[3]It was first invented by Mumford [Mum72], and later improved in [FC90] and [AN99]. We use the version in [AN99], and call it the AN construction.

not canonical when the degree of the polarization $d > 1$. From a degenerating family over the generic fiber, one gets two lattices $Y \subset X$ of index d, where d is the degree of $\mathcal{L}^*$, and a quadratic function $A : Y \to \mathbf{Z}$. Then one needs to choose a function φ over X that extends A. Different choices usually give different central fibers. For example, if one picks different extensions φ, one can get all central fibers in Figure 1.1. Besides the log geometry, one of the observations in [Ols08] is that there is a canonical choice of the extension φ: one simply requires the extension to be also quadratic over X. This choice gives I_3 for Example 1.2. However, from our point of view, this answer is unsatisfying for two things. First, the solution is implicit: if you want to know what the limit is, you need to run the machinery of the AN construction. There is no geometric explanation why the limit is canonical. Secondly, it is not obvious how to generalize the constructions to more general Calabi–Yau manifolds.

In this paper, we use mirror symmetry to give a different construction of Olsson's compactification. More precisely, we construct a compactification $\overline{\mathscr{A}}^m_{g,\delta}$ of the moduli space of polarized abelian varieties $\mathscr{A}_{g,\delta}$, which is isomorphic to the connected component $\overline{\mathscr{A}}_{g,\delta}[1/d]$ of $\overline{\mathscr{A}}_{g,d}$ (Proposition 4.64) over a field of characteristic zero. Our mirror symmetry approach is expected to work for more general Calabi–Yau manifolds. Moreover, we give an alternative way of constructing the central fiber. We show that the canonical central fiber can be obtained by using KSBA stable pairs. This is a little surprising because a stable pair is a scheme plus a divisor Θ^* from the linear system $|\mathcal{L}^*|$. People usually do not expect that there is a natural choice of divisors Θ^* when $\mathcal{L}^*$

has higher degree.

Here is our alternative interpretation of the extension for Example 1.2. The local monodromy of $\pi_1(\Delta^*)$ determines the space of vanishing cycles, an isotropic subspace U of $\mathrm{H}_1(X_t, \mathbf{R})$, where X_t is a general fiber. According to the classical theory of theta functions, U determines a finite collection of basis of $\mathrm{H}^0(X_t, \mathcal{L}_t)$, and each basis consists of theta functions ([BL04] Theorem 3.2.7)[4]. For example, the set $\{X, Y, Z\}$ is one such basis ([BL04] Exercise 7.7 (8)). The zero locus of the sum $X + Y + Z$ is a divisor Θ^* on $\mathscr{X}^*$ of degree 3, and $(\mathscr{X}^*, \Theta^*)$ is a stable curve with three marked points, an object in $\mathscr{M}_{1,3}$. Use the stable reduction theorem for the Deligne–Mumford's compactification $\overline{\mathscr{M}}_{1,3}$, we get the pair $(\mathscr{X}, \Theta)$ over Δ. Now if we forget about Θ, the underlying polarized family $(\mathscr{X}, \mathcal{O}(\Theta))$ is canonical, independent of the initial choice of Θ^*. Figure 1.2 is the picture of the central fiber I_3. Notice that the two dashed lines are different choices of divisors, but both pairs are stable: each irreducible component has a marked point in the smooth locus.

Now the generalization of $\overline{\mathscr{M}}_{1,d}$ for higher dimensions is $\overline{\mathscr{AP}}_{g,d}$, the moduli of KSBA stable pairs with actions of semi-abelian varieties $(\mathscr{X}, G, \Theta, \varrho)$ ([Ale96], [Ale02]). Our key observation is that, for abelian varieties, although one gets different pairs $(\mathscr{X}, \Theta)$ for different choices of Θ^* from S, the underlying polarized schemes $(\mathscr{X}, \mathcal{O}(\Theta))$ are all isomorphic, and the isomorphism class is the limit produced by Olsson's compactification $\overline{\mathscr{A}}_{g,d}$. Moreover, for abelian

[4]We will define this set in terms of the representation of the theta group. The set of the divisor will be denoted by $S(K_2)$.

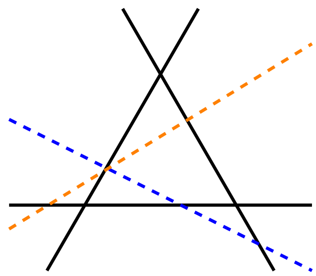

Figure 1.2: The Canonical Central Fiber With Different Choices of Divisors.

varieties, the set S can be defined in terms of the local monodromy and the representation of Mumford's theta group $G(M)$. Thus we can generalize the set S to an arbitrary degeneration, not just a maximal degeneration. We denote it by $S(K_2)$ (Definition 5.10), and call it a balanced set. Our main result is Theorem 5.13:

Theorem 1.4. *Suppose S is a normal local scheme over $\mathbf{C}$, (more precisely, see Assumption 5.8). Let $(\mathcal{X}, G, \mathcal{L}, \varrho)/S$ be a polarized stable semiabelic scheme over S, with the generic fiber abelian. Over the generic point η, the balanced set of divisors $S(K_2)$ is defined by the local monodromy. Then $(\mathcal{X}, G, \mathcal{L}, \varrho)$ is the pull-back of the AN family along a unique morphism $g : S \to \overline{\mathscr{A}}^{m}_{g,\delta}$ if and only if the group scheme $G(M)$ can be extended over S (thus $S(K_2)$ is also extended over S), and for one (equivalently any) divisor Θ from $S(K_2)$, $(\mathcal{X}, G, \Theta, \varrho)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.*

The definition (Definition 5.10) of $S(K_2)$ is a little involved. We only sketch it here. For any polarized abelian variety $(X, \mathcal{L})$, the space of global sections $\mathrm{H}^0(X, \mathcal{L})$ is an irreducible representation S^* of a theta group $G(M)$.

The local monodromy picks up a subgroup $K_w < G(M)$: the subgroup that preserves the vanishing cycles. Then we can decompose $\mathrm{H}^0(X, \mathcal{L})$ into K_w-irreducible representations $\mathrm{H}^0(X, \mathcal{L}) = \oplus_{\alpha \in I} V_\alpha$, where the index set I is the quotient of $G(M)$ by K_w. Take an arbitrary nonzero divisor ϑ_0 from one of the irreducible component V_α, choose an arbitrary lift $\sigma : I \to G(M)$, and define a divisor Θ to be the zero locus of the section

$$\vartheta = \sum_{\alpha \in I} S^*_{\sigma(\alpha)} \vartheta_0.$$

The set $S(K_2)$ is defined to be the set of all divisors obtained this way. It only depends on the local monodromy. This is why we need the base to be local and normal. For technical reasons, we can only prove the statement over $\mathbf{C}$. We expect it to be true as long as there is no bad prime reduction.

Notice when the polarization is principal, the linear system $|\mathcal{L}|$ contains only one element Θ, and $S(K_2)$ is simply $\{\Theta\}$. This recovers Alexeev's compactification of $\mathscr{A}_g$ inside $\overline{\mathscr{A}\mathscr{P}}_g$, up to a normalization[5].

Our compactification $\overline{\mathscr{A}}^m_{g,\delta}$ is also constructed in a different way. Since the coarse moduli space $\mathcal{A}_{g,\delta}$ over $\mathbf{C}$ is an arithmetic quotient of a Hermitian symmetric domain, we desire a toroidal compactification. The problem with the toroidal compactification is that it depends on extra data: an admissible collection of fans $\{\Sigma(F)\}$, and a priori there are infinitely many choices of

[5]Although, in [Ale02], the main component of the coarse moduli space $\overline{\mathcal{AP}}_g$ is claimed to be isomorphic to the second Voronoi compactification, the proof is incomplete. The author thanks Iku Nakamura for pointing this out. See ([Nak] Subsection 14.3).

$\Sigma(F)$s. We apply mirror symmetry to obtain the collection of fans $\{\Sigma(F)\}$. For the main ideas, see Sect. 1.3.

Theorem 1.5. *[Propostion 3.55 and Theorem 3.56] The collection of fans $\widetilde{\Sigma} = \{\Sigma(F)\}$ induced from the Mori fans of the minimal models of the mirror families is an admissible collection. Thus it gives a projective toroidal compactification $\overline{\mathcal{A}}_{\Sigma}$ of the coarse moduli space $\mathcal{A}_{g,\delta}$ over $\mathbf{C}$.*

In the case of principal polarization, we get the interesting fact that the second Voronoi fan for the toroidal compactification of $\mathcal{A}_g$ is equal to the Mori fan of the mirror (Theorem 3.50). By using the collection of fans $\{\Sigma(F)\}$ and the AN construction, we construct the compactification $\overline{\mathscr{A}}^m_{g,\delta}$ as an algebraic stack.

Theorem 1.6. *[Theorem 4.61] Over $k = \mathbf{Z}[1/d, \zeta_M]$, we have a proper Deligne-Mumford $\overline{\mathscr{A}}^m_{g,\delta}$ that contains $\mathscr{A}_{g,\delta}$ as a dense open substack. The coarse moduli space over $\mathbf{C}$ is the toroidal compactificaiton $\overline{\mathcal{A}}_{\Sigma}$. There is an AN family $(\mathcal{X}, G, \mathcal{L}, \varrho)$ extending the universal family over $\mathscr{A}_{g,\delta}$.*

The case of moduli space of toric pairs is much easier than the case of $\mathscr{A}_{g,\delta}$. We show that the compactification $\mathscr{K}_Q$ in [Ols08] can also be obtained by our mirror symmetry approach (Theorem 2.65). In particular, we prove that the secondary fan is equal to the Mori fan of the relative minimal models of the mirror family (Theorem 2.36).

1.3 The Motivation

Mirror symmetry is a phenomenon of deep, and in general spectacular equivalences between geometries on a pair of spaces X and Y. For example, the symplectic geometry of a Calabi–Yau manifold X is equivalent to the complex geometry of the mirror Calabi–Yau Y. The formulation is highly non-trivial. First the equivalence is between two derived categories or their A_∞ enhancements, and depends on the large complex structure limit[6]. Furthermore, the equivalence is only well-understood in a few special cases. For example, when the SYZ fibrations admit no singular fibers, the equivalence can be realized as the Fourier–Mukai transform. In general, it is not clear how to deal with the singular fibers. A perturbative approach taken by Gross and Siebert [GS06] is to understand the quantum correction in a formal neighborhood of the large complex structure limit, where the singular fibers support in a locus of codimension 2. We explain the basic philosophy, which involves only a small part of mirror symmetry, in this section. The philosophy is for general Calabi–Yau manifolds X. The rest of the thesis only deals with the cases where there is no singular fiber: the toric variety and the abelian variety.

Mirror symmetry helps us in two different ways. First, it suggests the canonical limit may be produced in the moduli of stable pairs. Thanks to the progress of the minimal model program, if the family $\mathcal{X}^*$ is equipped with an

[6] If one chooses a different a large complex structure limit of X, then the mirror space Y would be different.

additional datum: an ample divisor Θ^*, such that $(\mathcal{X}^*, \epsilon\Theta^*)$[7] is a stable pair in the sense of KSBA[8], then we have a unique extension of the pair $(\mathcal{X}, \epsilon\Theta)$ over the disk Δ in the moduli of stable pairs. However, this extension $(\mathcal{X}, \Theta)$ depends on the choice of Θ^*. In general, if we choose a different Θ^*, we will get a different underlying space $\mathcal{X}$.

Mirror symmetry predicts that, near a large complex structure limit, the polarized Calabi–Yau varieties $(\mathcal{X}^*, \mathcal{L}^*)$ are "secretly" stable pairs: there is a canonical basis $\vartheta_1, \ldots, \vartheta_d\}$ of the space of global sections $\mathrm{H}^0(\mathcal{X}^*, \mathcal{L}^*)$ given by the mirror maps. For K3 surfaces, this is Tyurin's conjecture([Tyu] Page 36. Remark), and is proved in [GHK]. Naturally, one can pick a divisor Θ^* defined by the sum $\sum_i \vartheta_i$ and take the limit (X_0, Θ_0) of $(\mathcal{X}^*, \Theta^*)$ in the proper moduli space of KSBA stable pairs. This produces a central fiber if we forget about the divisor Θ_0 in the pair. To be honest, the canonical basis is not completely canonical, there are some choices involved (The choice of a lagrangian in the mirror). When $\mathcal{X}^*$ is isomorphic to a family of abelian varieties, the canonical basis consists of theta functions produced from the representations of the theta groups. That is how we get the finite balanced set $S(K_2)$ near the 0-cusp. As is explained in Sect. 1.2, while different choices give rise to different pairs $(X_0, \Theta_0))$, the underlying polarized varieties $(X_0, \mathcal{O}(\Theta_0)$ are all isomorphic (Theorem 5.13). This is a possible way to produce the

[7] $\epsilon \ll 1$ is a sufficiently small positive number.

[8] KSBA is the short for Kollár–Shepherd-Barron–Alexeev. For the definition of a stable pair, see [Ale96].

canonical limit for Question 1.1.

Secondly, mirror symmetry provides canonical partial compactifications of the moduli space of Calabi–Yau manifolds $\mathcal{M}$ near large complex structure limits[9] by using Mori fans of the minimal models of the mirror families. This idea goes back to [Mor93]. According to the prediction of mirror symmetry [Mor93], near the large complex structure limits, $\mathcal{M}$ is a group quotient of a tube domain $\Gamma\backslash\mathcal{D}$. The picture is as follows. Let $\mathcal{C} \subset \mathbb{L}^*_{\mathbf{R}} = \mathbb{L}^* \otimes_{\mathbf{Z}} \mathbf{R}$ be an open strongly convex cone, with $\mathbb{L}^*$ a finitely generated free abelian group acting on $\mathbb{L}^*_{\mathbf{C}}$ by translation. The complex domain $\mathcal{D} := \mathbb{L}^*_{\mathbf{R}} \times i\mathcal{C} \subset \mathbb{L}^*_{\mathbf{C}}$ is called a tube domain. Let $\Gamma \subset \mathrm{Aff}(\mathbb{L}^*)$[10] be a discrete group that contains the translations $\mathbb{L}^*$ and preserves the cone $\mathcal{C}$. Therefore Γ is acting on $\mathcal{D}$. We regard the quotient $\Gamma\backslash\mathcal{D}$ as a two-step quotient. We first quotient out $\mathbb{L}^*$ and then quotient out $\overline{\Gamma} := \Gamma/\mathbb{L}^*$. Then $\mathbb{L}^*\backslash\mathcal{D}$ is contained in the algebraic torus $T_{\mathbb{L}^*} := \mathbb{L}^*\backslash\mathbb{L}^*_{\mathbf{C}}$. Given a fan Σ supported on the rational closure of the cone $\mathcal{C}$[11], we get a partial compactification of $\mathbb{L}^*\backslash\mathcal{D}$ by taking its closure in the toric variety $T_{\mathbb{L}^*} \subset X_\Sigma$. If the fan Σ is further $\overline{\Gamma}$-admissible, we can take the quotient and get a partial compactification of $\Gamma\backslash\mathcal{D}$. The amazing thing is that mirror symmetry provides a natural $\overline{\Gamma}$-admissible fan Σ. There is a 1-parameter mirror family of Calabi–Yau varieties $\mathcal{Y}$ with a general fiber Y, such

[9] We call them 0-cusps when $\mathcal{M}$ is an arithmetic quotient of the Hermitian symmetric domain.

[10] The group of affine transformations of $\mathbb{L}^*$.

[11] A fan whose support is the rational closure of $\mathcal{C}$ is a decomposition of the rational closure of $\mathcal{C}$ into rational polyhedral cones glued along faces.

that $\mathcal{C}$ is equal to the Kähler cone of Y, $\mathbb{L}^* = H^{1,1}(Y, \mathbf{R}) \cap H^2(Y, \mathbf{Z})/(\text{torsion})$, and Γ a subgroup of $\mathbb{L}^* \rtimes \mathrm{Aut}(Y)$ of finite index. Therefore, we look for a fan Σ supported on the Kähler cone of the mirror Y. The key idea from [GHK] is that we should consider the Mori fan of the relative minimal models of the 1-parameter family $\mathcal{Y}$. The Mori fan is defined in terms of the Stein factorizations of all birational contractions, and is independent of the choice of the relative minimal model. Therefore, we say that the Mori fan is canonical. There is also a canonical section along which we can pull back the Mori fan to the Käher cone of Y. When $\mathcal{M}$ is an arithmetic quotient of a Hermitian symmetric domain, it is straightforward to check if these partial compactifications can be glued together to give a global compactification of $\mathcal{M}$.

While the philosophy is highly conjectural for general Calabi–Yau manifolds, we show that it works well for two cases concerned in this thesis. Note that when the moduli space has more than one 0-cusp, the compatibility issue in the construction of a toroidal compactification is nontrivial. All the toroidal compactifications known to the author assumes that there is only one 0-cusp. Therefore the success of our approach here is encouraging. Moreover, one of the projects of Gross–Hacking–Keel ([GHK]) is to compactify the moduli of polarized K3 surfaces by using similar ideas. We expect that it should work for more general Calabi–Yau manifolds.

1.4 Outline

Chap. 2 is devoted to the case of toric pairs. The moduli space of stable toric paris has been studied in both [Ale02] and [Ols08]. Here, we show that it can also be constructed by using the combinatoric data from the mirror family. The main theorem for this chapter is Theorem 2.36 in Sect. 2.2.2 and an explicit construction (Theorem 2.65) of the stack in Sect. 2.3.

In Chap. 3 we construct the toroidal compactification. Sect. 3.1.1 and Sect. 3.1.2 is an overview of the construction of the moduli spaces over complex numbers and the cones $\mathcal{C}(F)$ needed in the toroidal compactification. In order to deal with the complicated group actions for higher degree polarizations, we interpret them as moduli spaces of polarized tropical abelian varieties in Sect. 3.1.3. This is Proposition 3.31. In Sect. 3.2, we study the mirror symmetry for abelian varieties. We construct the mirror family $\mathcal{Y}^*/\Delta^*$ and its minimal models $\mathcal{Y}_{\mathscr{P}}/\Delta$. The goal is to get the collection of fans $\widetilde{\Sigma} = \{\Sigma(F)\}$ (Definition 3.54) for the toroidal compactification from the Mori fans of $\mathcal{Y}_{\mathscr{P}}/\Delta$ through a linear section (Proposition 3.52). Sect. 3.3 is for the construction of the toroidal compactification over complex numbers. We prove that the collection of fans $\widetilde{\Sigma}$ gives a toroidal compactification (Proposition 3.55).

Once we get the correct fans, it is actually not difficult to construct the compactification as an algebraic stack with families extending the universal family over it. However this is very technical, and Chapter. 4 takes up a big part of the paper. We generalize the AN construction to a complete local base (Sect. 4.1), get the standard degeneration data compatible with our toroidal compactification (Sect. 4.2). Following the procedures in [FC90], we construct the algebraic stack $\overline{\mathscr{A}}^m_{g,\delta}$ (Theorem 4.61) in Sect. 4.4. In particular we give explicit constructions of the local charts over the formal base and show how to glue them in Theorem 4.46 and Theorem 4.52.

Finally, in Chap. 5 we explain the geometric characterization of the extended family over the boundary in terms of KSBA stable pairs. The main theorem is Theorem 5.13.

1.5 Conventions and Notations

Let X be a normal scheme. $\mathrm{Pic}(X)$ is the Picard group. $\mathrm{Cl}(X)$ is the Weil divisor class group. $\mathrm{CaCl}(X)$ is the Cartier divisor class group. If X is integral, $\mathrm{Pic}(X) \cong \mathrm{CaCl}(X)$. $\mathrm{NS}(X)$ is the Neron-Severi group. $\mathrm{N}_1(X)$ is the $\mathbf{R}$-vector space of curves modulo numerical equivalence. $\mathrm{Eff}(X)$ is the effective cone in $\mathrm{N}^1(X)$. $\mathrm{NE}(X)$ is the effective cone in $\mathrm{N}_1(X)$. $\mathrm{Mov}(X)$ is the movable cone in $\mathrm{NS}(X)_{\mathbf{R}}$. We add the subscript $\mathbf{R}$ if we talk about $\mathbf{R}$ classes. Intersection of curves and divisors gives a perfect pairing $\mathrm{N}_1(X) \times \mathrm{NS}(X)_{\mathbf{R}} \to \mathbf{R}$.

A function f is *convex* if $f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$. Notice that it is different from the usual convention in the literature for toric geometry. For a Cartier divisor $\{m_\sigma\}$ on a toric variety, we define the associated piecewise linear function to be $\varphi = -m_\sigma$. For a Weil divisor $D = \sum a_\omega D_\omega$, the associated function is $\psi(\omega) = a_\omega$.

If N is a $\mathbf{Z}$-module $N_k := N \otimes_{\mathbf{Z}} k$ for a field k. An integral structure on an affine space $\mathbf{A}_k$ over k is an isomorphism $\mathbf{A}_k \cong N_k$ for some lattice N, and we regard N as a lattice inside $\mathbf{A}_k$.

A polytope Q is the convex hull of finitely many points in an affine space, and thus is always bounded. If the affine space has an integral structure, and all the points can be chosen from the lattice, then Q is called a lattice polytope. A polyhedron is the intersection of (not necessarily finitely many) closed half spaces. If a polyhedron is finite and bounded, then it is a polytope. This a well-known theorem, but not obvious. See ([CLS11] p. 63). The definition of lattice polyhedron is ([CLS11] Definition 7.1.3.). For an arbitrary subset S of an affine space, the cone generated by S is denoted by $C(S)$, and the convex hull is denoted by $\mathrm{Conv}(S)$.

For toric varieties, we use the definitions and notations in [CLS11] unless specified otherwise. Let N be a lattice. A rational polyhedra cone (loc. cit. Definition 1.2.14.) is usually denoted by $\sigma \subset N_{\mathbf{R}}$, and the complement of all the proper faces is denoted by $\overset{\circ}{\sigma}$. However, for a fan used in the toroidal compactification, we define the cones σ to be the interior of the rational polyhedra cone, following the convention of [FC90]. Our main example is the second Voronoi fan (Definition 3.49), where $C(\mathscr{P})$ is defined to be the closure of the cone $\sigma(\mathscr{P}) = \overset{\circ}{\sigma} (\mathscr{P})$ for a Delaunay decomposition $\mathscr{P}$.

The dual lattice of N is M, and the dual cone (always defined to be closed) in $M_{\mathbf{R}}$ is $\sigma^\vee$. The toric monoid $S_\sigma := \sigma^\vee \cap M$. If Σ is a fan in N, the associated toric variety is denoted by X_Σ. The torus with the cocharacter group N is denoted by T_N.

The set of nonnegative integers is denoted by $\mathbf{N}$. In particular, $0 \in \mathbf{N}$. If M is a monoid, then M^{gp} denotes the group associated to the monoid, and M^{sat} is the saturation.

Aff stands for the vector space of affine functions, PA, the space of piecewise affine functions, PL, the space of piecewise linear functions. Usually there are other decorations in the notations.

Let d be an integer, and δ be a sequence $(\delta_1, \dots, \delta_g)$ of positive integers $\delta_1 | \delta_2 \dots | \delta_g$ such that $\prod_{i=1}^{g} \delta_i = d$. We also use δ to denote the diagonal matrix

$$\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_g \end{pmatrix}.$$

Define the finite group scheme over $\operatorname{Spec} \mathbf{Z}$

$$K(\delta) := \prod_{i=1}^{g} \mu_{\delta_i} \times \mathbf{Z}/\delta_i \mathbf{Z}.$$

The base change of $K(\delta)$ is either denoted by $K(\delta)$ or $K(\delta)_T$. There is a natural nondegenerate alternating pairing over $K(\delta)$,

$$\begin{aligned} e_\delta : K(\delta) \times K(\delta) &\longrightarrow \mathbb{G}_m, \\ \big((\zeta_i, n_i)_{i=1,\dots,g}, (\eta_j, m_j)_{j=1,\dots,g}\big) &\longmapsto \prod_{i=1}^{g} \frac{\zeta_i(m_i)}{\eta_i(n_i)}. \end{aligned}$$

We call e_δ the standard pairing over $K(\delta)$. Define $\mathcal{H}(\delta)$ to be the central extension of $K(\delta)$ by $\mathbb{G}_m$ inducing the standard pairing e_δ. In other words, for each pair of elements $g, h \in K(\delta)$, consider arbitrary lifts $\tilde{g}, \tilde{h}$ to the central extension. Then the commutator $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$ is an element in $\mathbb{G}_m$, and is independent of the choice of the lifts. This defines an alternating pairing over $K(\delta)$, and we require that this alternating pairing is the standard pairing e_δ.

The category of schemes over a base scheme S is denoted by Sch/S. If k is the base ring, the Cartesian product $X \times_k Y$ is written as $X \times Y$. We use the font $\mathscr{M}$ for a stack, and $\mathcal{M}$ for its coarse moduli space if it exists. An algebraic stack is usually represented as a groupoid $R \rightrightarrows U$ in Sch/S. By a pre-equivalence relation, we mean $R \to U \times_S U$ is not necessarily a monomorphism.

The upper half plane $\mathbf{H} = \{\tau \in \mathbf{C} : \Im\tau > 0\}$. Let Δ denote an open disk with a radius sufficiently small, and Δ^* be the punctured disk.

Definition 1.7. The moduli stack $\mathscr{A}_{g,d}$ is the stack over the fppf site of schemes, such that, for any scheme S, $\mathscr{A}_{g,d}(S)$ is the groupoid of $(\mathcal{X}, \lambda)/S$, where $\mathcal{X}$ is an abelian scheme over S of relative dimension g, and $\lambda : \mathcal{X} \to \mathcal{X}^t$ is an isogeny of degree d^2, that is locally induced by an ample invertible sheaf. The degree of the polarization is defined to be d.

Definition 1.8. The moduli stack $\mathscr{A}_{g,\delta}$ is the full subcategory of $\mathscr{A}_{g,d}$ consisting of the objects $(\mathcal{X}, \lambda)/S$ such that there exists a faithfully flat $T \to S$ such that $\ker(\lambda)_T \cong K(\delta)_T$.

Both $\mathscr{A}_{g,d}$ and $\mathscr{A}_{g,\delta}$ are Deligne–Mumford stacks over $\mathbf{Z}$. For basic properties and remarks, see [dJ93].

We need the following definitions of moduli spaces. The moduli of stable toric pairs, $\mathscr{T}\mathscr{P}^{\mathrm{fr}}[Q]$, the moduli stack of abelic pairs of degree d, $\mathscr{A}\mathscr{P}_{g,d}$, and the moduli stack of stable semi-abelic pairs of degree d, $\overline{\mathscr{A}\mathscr{P}}_{g,d}$ are defined in [Ale02]. The moduli spaces $\mathscr{K}_Q$, $\mathscr{T}_{g,d}$, $\overline{\mathscr{T}}_{g,d}$, and $\overline{\mathscr{A}}_{g,d}$ are defined in [Ols08].

Chapter 2

Moduli of Stable Toric Pairs

2.1 The Constructions of Families

2.1.1 The Cone Constructions

Given a vector space V, we can forget about the zero vector and get an affine space. This is the forgetful functor (from linear to affine) $\mathbb{R}$. The left adjoint to $\mathbb{R}$ is the embedding of an affine space into a vector space of 1-dimension higher as the hyperplane of height 1. In this section, we introduce canonical constructions of similar left adjoints $\mathbb{L}$, S, C, which turn affine structures into additive (or linear) structures.

Definition 2.1. Let k be a field, and $\overline{V}$ be an affine space over some k-vector space V. Define a k-linear structure over the set

$$\mathbb{L}(\overline{V}) := \{(t, q) : q \in \overline{V}, t \in k\backslash\{0\}\} \cup \{(0, v) : v \in V\}.$$

The addition is defined to be

$$
\begin{aligned}
(t,p)+(s,q) &= (t+s, \frac{t}{t+s}p+\frac{s}{t+s}q), \text{ if } t\neq 0, s\neq 0,\\
(t,p)+(0,v) &= (t, \frac{1}{t}v+p),\\
(t,p)+(s,q) &= (0, t(p-q)), \text{ if } t+s=0, t\neq 0,\\
(0,u)+(0,v) &= (0,u+v).
\end{aligned}
$$

The scalar multiplication is

$$
\begin{aligned}
s(t,q) &= (st,q), \text{ if } s\neq 0, t\neq 0\\
s(t,q) &= 0, \text{ if } s=0, t\neq 0,\\
s(t,v) &= (0,sv), \text{ if } t=0.
\end{aligned}
$$

Define the degree map

$$
\begin{aligned}
\deg : \mathbb{L}(\overline{V}) &\longrightarrow k\\
(t,q) &\longmapsto t
\end{aligned}
$$

The kernel of deg is V. We always identify $q\in\overline{V}$ with $(q,1)\in\mathbb{L}(\overline{V})$. Therefore, we regard $\overline{V}$ as the hyperplane of $\mathbb{L}(\overline{V})$ of height 1. This embedding induces a bijection between the set of linear functions on $\mathbb{L}(\overline{V})$, and the set of affine functions on $\overline{V}$. For simplicity, $\mathbb{L}(\overline{V})$ is also denoted by $\mathbb{V}_k$ or $\mathbb{V}$.

Let $k=\mathbf{R}$. Let X be a free abelian group, and $X\cong\mathbf{Z}^g$. Let $\overline{X}$ be an X-torsor. For a positive integer n, define

$$
\overline{X}(1/n) := \Big\{ q\in\overline{X}_{\mathbf{Q}} : q=\sum_i \frac{a_i}{n}q_i, \text{ for } q_i\in\overline{X}, a_i\in\mathbf{Z}, \sum_i a_i = n \Big\}.
$$

Regard $\overline{X}$ to be contained in the subset of $\mathbb{L}(\overline{X}_{\mathbf{R}})$ of degree 1. Define a free abelian group $\mathbb{L}(\overline{X})$ to be the subgroup of $\mathbb{L}(\overline{X}_{\mathbf{R}})$ generated by $\overline{X}$,

$$\mathbb{L}(\overline{X}) := \Big\{(n,q) \in \mathbf{Z}\backslash\{0\} \times \overline{X}_{\mathbf{Q}} : q \in \overline{X}(1/n)\Big\} \cup (\{0\} \times X).$$

As an abelian group $\mathbb{L}(\overline{X}) \cong \mathbf{Z}^{g+1}$. It has a grading by $\deg : \mathbb{L}(\overline{X}) \to \mathbf{Z}$. $\mathbb{L}(\overline{X})$ is also denoted by $\mathbb{X}$.

If φ is an affine function on $\overline{X}_{\mathbf{R}}$, the linear part is the differential and is denoted by $\mathrm{D}\varphi$,

$$\mathrm{D}\varphi(p-q) = \varphi(p) - \varphi(q).$$

The affine function φ is called integral if it takes integer values on $\overline{X}$. This implies that $\mathrm{D}\varphi$ is also integral. Denote the set of integral affine functions by $Aff(\overline{X}, \mathbf{Z})$. $\mathbb{X}$ is canonically isomorphic to the dual $Aff(\overline{X}, \mathbf{Z})^*$ via

$$\begin{aligned}(n,q)(\varphi) &= n\varphi(q), n \neq 0\\ (0,q)(\varphi) &= \mathrm{D}\varphi(q).\end{aligned}$$

The isomorphism above maps $q \in \overline{X}$ to the evaluation map $\mathrm{ev}_q \in Aff(\overline{X}, \mathbf{Z})^*$ and thus is denoted by ev.

In both cases, $\mathbb{L}$ are functors. For example, if $\varphi : \overline{X} \to \overline{X'}$ is affine (or piecewise affine), the corresponding additive (piecewise additive) map is

$$\begin{aligned}\mathbb{L}(\varphi) : \mathbb{L}(\overline{X}) &\longrightarrow \mathbb{L}(\overline{X'}),\\ (n,q) &\longmapsto \begin{cases} n\varphi(q) & \text{if } n \neq 0\\ \mathrm{D}\varphi(q) & \text{if } n = 0.\end{cases}\end{aligned}$$

Let Q be a lattice polytope in $\overline{X}_{\mathbf{R}}$. Define $Q(\frac{1}{n}) := \{q \in Q \cap \overline{X}(\frac{1}{n})\}$, and $Q(\mathbf{Q}) := \coprod_{n>0} Q(\frac{1}{n})$. Identify $Q(\mathbf{Q}) \cup \{0\}$ with a subset $S(Q) \subset \mathbb{X}$.

Definition 2.2. $S(Q)$ is defined to be a graded monoid, whose underlying set is,

$$S(Q) := \Big\{(n,q) : q \in Q(1/n), n \in \mathbf{N}\backslash\{0\}\Big\} \cup \{0\},$$

with the addition and grading induced from those on $\mathbb{X}$.

Remark 2.3. $Q(\mathbf{Z}) \cong \deg^{-1}(1)$. If Q is of full dimension, the associated group of $S(Q)$ is $\mathbb{X}$. $S(Q)$ is a toric monoid.

This construction can be generalized to an unbounded lattice polyhedron Q. In this case we have to consider the infinite direction. Define the degree 0 part to be

$$S(Q)_0 := \{(0,\alpha) : \alpha \in X, Q + \alpha \subset Q\}.$$

Define $(0,\alpha) + (n,q) = (n, \frac{\alpha}{n} + q)$ if $n \neq 0$. Although $S(Q)$ is not fine anymore, it is finitely generated over $S(Q)_0$.

Remark 2.4. We will use $q \in Q(\mathbf{Q})$ to represent an element of $S(Q)$ if there is no confusion. If we have chosen an origin in $\overline{X}$, we can identify $\mathbb{X}$ with $\mathbb{Z} \oplus X$. In this case, we use (n,α), with $\alpha \in X$ to represent an element in $\mathbb{X}$ or $S(Q)$. The addition is the usual addition: $(m,\alpha) + (n,\beta) = (m+n, \alpha+\beta)$.

Regard $S(Q)$ as a subset of $\mathbb{X}_{\mathbf{R}}$. The convex hull generated by $S(Q)$ in $\mathbb{X}_{\mathbf{R}}$ is a cone, and is denoted by $C(Q)$. Since Q is a lattice polyhedron,

$C(Q)$ is a rational polyhedral cone. If Q is a lattice polytope, $C(Q)$ is strongly convex. We have

$$S(Q) = C(Q) \cap \mathbb{X}, \text{ and } Q = C(Q) \cap \mathbb{X}_{\mathbf{R},1}.$$

Definition 2.5. Let V be a $\mathbf{R}$ vector space. For any (piecewise) affine function $\varphi : Q \to V$, we associate a (piecewise) linear function $\tilde{\varphi} : C(Q) \to V$ by

$$\tilde{\varphi}(q) := \begin{cases} \deg(q)\varphi(q/\deg(q)) & \text{if } \deg(q) \neq 0 \\ \mathrm{D}\varphi(q) & \text{if } \deg(q) = 0 \end{cases}$$

On the other side, given a (piecewise) linear function $\tilde{\varphi} : C(Q) \to V$, the restriction to Q is an (piecewise) affine function. So there is a bijection between the set of linear functions on $C(Q)$ and the set of affine functions on Q. Define $\mathbb{R}$ to be the forgetful functor from the category of vector spaces to the category of affine spaces. A special case of the bijection $\varphi \mapsto \tilde{\varphi}$ is

Corollary 2.6. Let U be an affine space, and V a vector space over the same field. There is a natural isomorphism

$$\mathrm{Hom}(U, \mathbb{R}(V)) \cong \mathrm{Hom}(\mathbb{L}(U), V).$$

In particular, $\mathbb{L}$ is the left adjoint to the forgetful functor $\mathbb{R}$.

Definition 2.7. A piecewise affine (resp. linear) function φ (resp. $\tilde{\varphi}$) is called integral if it is integral on every top-dimensional affine (resp. linear) domain.

Remark 2.8. A piecewise affine map φ is integral if and only if $\tilde{\varphi}$ is integral. This integrality is stronger than saying φ takes integral values on each integral point. It also requires the slopes to be integral.

2.1.2 The Toric Constructions

In this section, k is an arbitrary commutative noetherian ring. We abuse the terminology "toric variety" even when it is not over a field. Let Q be a full dimensional lattice polytope in $\overline{X}_{\mathbf{R}}$. We recall the constructions in [Ale02], and fix our notations. Define a graded k-algebra R. Let $R_0 = k$. For any $n \in \mathbf{N}\backslash\{0\}$, let R_n be a free R_0-module generated by X^q for $q \in S(Q)_n$. $R := \bigoplus R_n = k[S(Q)]$ is a graded k-algebra given by the monoid $S(Q)$. Define the projective scheme $X_Q := \operatorname{Proj} R$ over k. X_Q is a projective toric variety with an ample line bundle $\mathcal{L} = \mathcal{O}(1)$.

Define the tori $\mathbb{T} := \operatorname{Spec} k[\mathbb{X}]$ and $T := \operatorname{Spec} k[X]$. The exact sequence

$$0 \longrightarrow X \longrightarrow \mathbb{X} \xrightarrow{\deg} \mathbf{Z} \longrightarrow 0$$

induces an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{T} \longrightarrow T \longrightarrow 1.$$

The injection $S(Q) \to \mathbb{X}$ defines a $\mathbb{T}$-action ϱ on the graded ring $k[S(Q)]$. Since the exact sequence splits, ϱ induces a T-linearization of $\mathcal{L}$. Pick a section ϑ of $\Gamma(X_Q, \mathcal{L})$. It is decomposed into eigenvectors under the action of T.

$$\vartheta = \sum_{\omega \in Q(\mathbf{Z})} c(\omega)\mathrm{X}^{\omega}.$$

The section ϑ is called *stable* for X_Q if $c(\omega) \in k^*$ for all vertices ω of Q. Take the divisor $\Theta := (\vartheta)_0$. We get $(X_Q, \mathcal{L}, \Theta, \varrho)$. By construction, it is a stable toric pair.

More generally, given an integral polyhedral paving $\mathscr{P}$ of Q (we also denote a polytope Q with a paving $\mathscr{P}$ by $\underline{Q}$), we define a broken toric variety $X_{\mathscr{P}}$. $\underline{Q}$ is a complex of lattice polytopes referenced by X. Then $X_{\mathscr{P}} := P[\underline{Q}, \tau]$ for τ trivial. Here $P[\underline{Q}, \tau]$ is defined in ([Ale02] Definition 2.4.2). The projective scheme $X_{\mathscr{P}}$ is equipped with an ample line bundle $\mathcal{L}$, and a T-linearization ϱ. $(X_{\mathscr{P}}, \mathcal{L}, \varrho)$ is a polarized stable toric scheme (loc. cit. Theorem 2.4.4). Pick a section $\vartheta \in \Gamma(X_{\mathscr{P}}, \mathcal{L})$

$$\vartheta = \sum_{\omega \in Q(\mathbf{Z})} c_\omega \mathrm{X}^\omega$$

Take the divisor $\Theta = (\vartheta)_0$. Over an algebraically closed field, a pair $(X, \mathcal{L}, \varrho, \Theta)$ is called a *stable toric pair* if X is a projective stable toric scheme and Θ is an effective ample Cartier divisor which does not contain any T-orbit over each geometric point. This is true if and only if $c(\omega)$ is invertible for each vertex ω of the paving $\mathscr{P}$ ([Ale02] Lemma 2.6.1). In this case, we call the section ϑ *stable* for $X_{\mathscr{P}}$.

The degeneration of the polarized stable toric schemes corresponds to refining the paving $\mathscr{P}$. Here is an example. See [GS09] for details.

Example 2.9. *Let Q be a full dimensional lattice polytope in $\overline{X}_{\mathbf{R}}$. To construct a degeneration of X_Q over $\mathbf{A}^1$, we need extra data $(\mathscr{P}, \varphi)$. Here $\mathscr{P}$ is a paving of Q, and φ is a $\mathscr{P}$-piecewise affine, strictly convex function over Q. We call it integral if it is integral restricting to each affine domain. Let*

$$Q_\varphi := \{(\alpha, h) \in Q \times \mathbf{R} : h \geqslant \varphi(\alpha)\} \subset \overline{X}_{\mathbf{R}} \times \mathbf{R}.$$

Let $X_{Q_\varphi} := \operatorname{Proj} k[S(Q_\varphi)]$ be the scheme constructed from the unbounded polyhedron Q_φ. It comes with a natural toric projection $\pi : X_{Q_\varphi} \to \mathbf{A}^1$ because $R_0 = k[q]$, where $q = \mathrm{X}^{(0,1)}$. The central fiber $\pi^{-1}(0)$ is reduced if and only if φ is integral. In this case $\pi^{-1}(0)$ is the broken toric variety $X_{\mathscr{P}}$. The general fibers $\pi^{-1}(q), q \neq 0$, are X_Q. Therefore the data $(\mathscr{P}, \varphi)$ gives a degeneration of X_Q to $X_{\mathscr{P}}$. The total space X_{Q_φ} is also denoted by $\mathcal{X}_\varphi$.

Remark 2.10. Let φ and φ' be two $\mathscr{P}$-piecewise affine, strictly convex functions. $\varphi' - \varphi$ is an integral affine function if and only if $S(Q_\varphi)$ and $S(Q_{\varphi'})$ are isomorphic as monoids. In this case, $\mathcal{X}_\varphi$ and $\mathcal{X}_{\varphi'}$ are isomorphic over $\mathbf{A}^1$.

Therefore, if the linear space of $\mathscr{P}$-piecewise affine functions is denoted by $PA(\mathscr{P}, \mathbf{R})$, then the construction only depends on the data in $PA(\mathscr{P}, \mathbf{R})/Aff$. So the natural way is to define φ by its bending parameters. We use a collection of bending parameters to represent an element in the set $PA(\mathscr{P}, \mathbf{R})/Aff$.

We do this in a more general setting. Let P be a toric monoid, i.e. P is fine and saturated and the associated group P^{gp} is torsion free. Assume $\sigma_P := \operatorname{Conv}(P)$ in $P^{\mathrm{gp}}_{\mathbf{R}}$. σ_P is a polyhedral cone. Introduce a partial order on $P^{\mathrm{gp}}_{\mathbf{R}}$.

Definition 2.11. For $u, v \in P^{\mathrm{gp}}_{\mathbf{R}}$, we say u is P-*above* v, and denote it by $u \overset{P}{\geqslant} v$, if $u - v \in \sigma_P$. We say u is strictly P-above v if in addition, $u - v \in \sigma_P \backslash P^*_{\mathbf{R}}$.

Definition 2.12 (bending parameters)**.** For each codimension 1 cell $\rho \in \mathscr{P}$

contained in maximal cells $\sigma_+, \sigma_- \in \mathscr{P}$, we can write

$$\varphi|_{\sigma_+} - \varphi|_{\sigma_-} = n_\rho \otimes p_\rho,$$

where n_ρ is the unique primitive element that defines ρ and is positive on σ_+, and $p_\rho \in P^{\mathrm{gp}}_{\mathbf{R}}$ is called the *bending parameter* for the wall ρ.

Definition 2.13. A piecewise affine function φ is P*-convex* if for every codimension one cell $\rho \in \mathscr{P}$, $p_\rho \in P$. It is *strictly* P*-convex* if all $p_\rho \in P \backslash P^*$.

If φ is affine over Q, the bending parameters are all trivial. If $\varphi, \varphi' \in PA(\mathscr{P})$ have bending parameters $\{p_\rho\}$, $\{p'_\rho\}$ respectively, the bending parameters for $\varphi + \varphi'$ is $\{p_\rho + p'_\rho\}$. Therefore, an element in $PA(\mathscr{P})/Aff$ (with values in $P^{\mathrm{gp}}_{\mathbf{R}}$) is determined by the collection of bending parameters.

Let $Q \subset \overline{X}_{\mathbf{R}}$ be a full dimensional lattice polytope with integer points $I := Q(\mathbf{Z})$, and $\mathscr{P}$ be a paving. Fix a sharp toric monoid P. Define the trivial bundle $\mathbb{P} \to Q$

$$\mathbb{P} := Q \times P^{\mathrm{gp}}_{\mathbf{R}}.$$

Assume $\varphi : \underline{Q} \to P^{\mathrm{gp}}_{\mathbf{R}}$ is a $\mathscr{P}$-piecewise affine, P-convex, integral function. Define

$$Q_\varphi := \Big\{(\alpha, h) \in \mathbb{P} : h \overset{P}{\geqslant} \varphi(\alpha)\Big\}.$$

Define $R_\varphi := k[S(Q_\varphi)]$ and $S := \operatorname{Spec} k[P]$. Notice that $R_{\varphi,0} = k[P]$. R_φ is of finite type over $k[P]$. Define $\mathcal{X}_\varphi = \operatorname{Proj} R_\varphi$. We have $\pi : \mathcal{X}_\varphi \to S$. π is projective and of finite type. The line bundle $\mathcal{L} := \mathcal{O}(1)$ is π-ample. The $\mathbb{X}$-grading on $k[S(Q_\varphi)]$ induces an action ϱ of $\mathbb{T}$ on R_φ.

Notice that $P \to S(Q_\varphi)$ is an integral morphism of integral monoids. $S(Q_\varphi)^{\mathrm{gp}} = \mathbb{L}(\overline{X} \times P^{\mathrm{gp}}) \cong \mathbb{X} \times P^{\mathrm{gp}}$. Under this isomorphism, the morphism $P^{\mathrm{gp}} \to S(Q_\varphi)^{\mathrm{gp}}$ is the homomorphism

$$P^{\mathrm{gp}} \longrightarrow \mathbb{X} \times P^{\mathrm{gp}},$$
$$p \longmapsto (0, p).$$

Then the cokernel of $f^\flat : P \to S(Q_\varphi)$ is equal to the image of $S(Q_\varphi)$ in $S(Q_\varphi)^{\mathrm{gp}}/P^{\mathrm{gp}} = \mathbb{X}$, which is $S(Q)$. Denote the projection $S(Q_\varphi) \to S(Q)$ by b. Since P is sharp, by ([Ols08] Lemma 3.1.32), for any $q \in S(Q)$, there exists a unique element $\tilde{q} \in b^{-1}(q)$ such that $\tilde{q} \overset{P}{\leqslant} x$ for all $x \in b^{-1}(q)$. For any $q \in S(Q)$, define $\vartheta_q := \mathrm{X}^{\tilde{q}}$. By definition, for any $x \in b^{-1}(q)$, $x - \tilde{q} \in P$. Therefore

Proposition 2.14. *R_φ is a free $k[P]$-module generated by $\{\vartheta_q\}$ for all $q \in S(Q)$. In particular, the basis is parametrized by the rational points $Q(\mathbf{Q}) \cup \{0\}$.*

If φ is integral, then $\vartheta_q = \mathrm{X}^{(q,\varphi(q),n)}$. The multiplication rule is, for $\alpha \in Q(\frac{1}{n}), \beta \in Q(\frac{1}{m})$, and

$$\gamma = \frac{n}{n+m}\alpha + \frac{m}{n+m}\beta \in Q(\frac{1}{n+m}),$$

we have

$$\vartheta_\alpha \cdot \vartheta_\beta = \mathrm{X}^{n\varphi(\alpha)+m\varphi(\beta)-(n+m)\varphi(\gamma)}\vartheta_\gamma.$$

For φ, define a monoid $S(Q) \rtimes P$. As a set $S(Q) \rtimes P = S(Q) \times P$. The addition is defined by

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + n\varphi(\alpha) + m\varphi(\beta) - (n+m)\varphi(\gamma))$$

The conclusion is,

Proposition 2.15. *As monoids,*

$$S(Q_\varphi) \cong S(Q) \rtimes P.$$

Let $S = \operatorname{Spec} k[P]$ and $S' = \operatorname{Spec} k[P']$ be two affine toric varieties, and $f : S' \to S$ is a toric map induced by a homomorphism $f^\flat : P \to P'$. Then any P-convex piecewise affine function φ induces a P'-convex piecewise affine function φ' in the following way. Denote the paving of Q from φ by $\mathscr{P}$. φ defines a collection of pending parameters $\{p\}$ for $\mathscr{P}$. Define new bending parameters $\{p' = f^\flat \circ p\}$. They form a compatible collection of bending parameters with values in P'. Therefore, there exists a $\mathscr{P}$-piecewise affine function φ' which has bending parameters $\{p'\}$.

Corollary 2.16. The construction is functorial, i.e., the following diagram is cartesian.

$$\begin{tikzcd}
\mathcal{X}_{\varphi'} \arrow[r] \arrow[d] & \mathcal{X}_\varphi \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}$$

Proof. The morphism $\mathcal{X}_{\varphi'} \to \mathcal{X}_\varphi$ is induced from the map

$$S(Q) \rtimes P \to S(Q) \rtimes P'$$
$$(\alpha, p) \mapsto (\alpha, f^\flat(p))$$

□

Corollary 2.17 (Functorial)**.** Let S, S', P, P' and f be as above. Let φ be a P-convex piecewise affine function, and φ' be a P'-convex piecewise affine functions. Assume that $\psi = f^\flat\varphi - \varphi'$ takes values in $(P')^*$, the invertible elements in P', then we still have the pull back diagram

$$\begin{tikzcd} \mathcal{X}_{\varphi'} \arrow[r] \arrow[d] & \mathcal{X}_\varphi \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

Proof. We only need to check that $\mathcal{X}_{f^\flat\varphi} \cong \mathcal{X}_{\varphi'}$ over S'. The isomorphism is given by

$$S(Q) \rtimes P' \to S(Q) \rtimes P' \tag{2.1}$$

$$(\alpha, p) \mapsto (\alpha, p + \deg(\alpha)\psi(\alpha)). \tag{2.2}$$

It preserves the multiplication. It is an isomorphism because ψ takes values in $(P')^*$. □

Now choose $P' = \mathbf{N}$ and $S' = \mathbf{A}^1_k$. For any toric divisor over S with center in the toric boundary of S, we have a discrete valuation $v : P \to \mathbf{N}$. Use v as $f^\flat$. By Corollary 2.16, the pull–back $f^*\mathcal{X}_\varphi/\mathbf{A}^1$ is the 1-parameter family $\mathcal{X}_{\varphi'}/\mathbf{A}^1$ defined by the ordinary convex function $\varphi' = v \circ \varphi$ in Example 2.9. Assume the paving for φ' is $\mathscr{P}'$ which is coarser than $\mathscr{P}$. The generic fiber of $\mathcal{X}_{\varphi'}$ is the polarized toric variety $(X_Q, \mathcal{L})$ given by the lattice polytope Q. The central fiber of $\mathcal{X}_{\varphi'}$ is $(P, L)[\Delta', 1]$, where $(P, L)[\Delta', 1]$ is defined in ([Ale02] Definition 2.4.2) and Δ' is the complex obtained from the paving $\mathscr{P}'$. See

([Ale02] Lemma 2.8.4). We denote $P[\Delta', 1]$ by $X_{\mathscr{P}'}$. By ([Ale02] Corollary 2.4.5), $X_{\mathscr{P}'} = \varinjlim_{\sigma \in \mathscr{P}'} X_\sigma$.

Take a global section of $\mathcal{L}$

$$\vartheta := \sum_{\omega \in Q(\mathbf{Z})} c_\omega \vartheta_\omega, \tag{2.3}$$

for $c_\omega \in k[P]$. Define $\Theta := (\vartheta)_0$. Θ is a divisor of $\mathcal{X}_\varphi$, and is flat over S. The pull–back of Θ is stable for $\mathcal{X}_{\varphi'}$ if and only if the coefficients the residue of c_ω at the origin of $\mathbf{A}^1$ does not vanish for any $\omega \in \mathscr{P}' \cap I$ ([Ale02] Lemma 2.6.1). Therefore, we have

Proposition 2.18. *Let $(\mathcal{X}_\varphi, \mathcal{L}, \varrho)/S$ be constructed as above. If we choose ϑ such that c_ω does not vanish at any point $s \in S$ for any $\omega \in Q(\mathbf{Z}) \cap \mathscr{P}$, then $(\mathcal{X}, \Theta, \varrho)/S$ is a stable toric pair and is a degeneration of the pair (X_Q, Θ).*

The toric base $S = \operatorname{Spec} k[P]$ has a natural log structure $S^\dagger$ induced by the map $P \to k[P]$. Denote the log scheme $\operatorname{Spec} k$ with the trivial log structure by $\operatorname{Spec} k^\dagger$. $S^\dagger$ is log smooth over $\operatorname{Spec} k^\dagger$. See ([Ols08] Example 2.3.17). The natural map $S(Q) \rtimes P \to R_\varphi = k[S(Q) \rtimes P]$ induces a log structure on $\mathcal{X}_\varphi = \operatorname{Proj} R_\varphi$ as follows. The $\mathbb{G}_m$-action on $\operatorname{Spec} R_\varphi$ extends naturally to an action on the log scheme $\operatorname{Spec}(S(Q) \rtimes P \to R_\varphi)$ over $\operatorname{Spec} k[P]$. Since $\mathcal{X}_\varphi$ is the quotient $\operatorname{Spec}[R_\varphi] \backslash \{0\} / \mathbb{G}_m$, we obtain a log structure $\mathcal{X}_\varphi^\dagger$ by the descent theory of log structures. The morphism $f^\flat : P \to S(Q) \rtimes P$ induces a morphism, still denoted by $f^\flat$ between the associated log structures. The map $(f, f^\flat) : \mathcal{X}_\varphi^\dagger \to S^\dagger$ is a log morphism between log schemes by the following commutative diagram

$$\begin{tikzcd} S(Q) \rtimes P \arrow[r] & k[S(Q) \rtimes P] \\ P \arrow[u, "f^\flat"] \arrow[r] & k[P] \arrow[u, "f^\sharp"'] \end{tikzcd} .$$

By the criterion ([Ols08] Theorem 2.3.16), the log morphism $(f, f^\flat)$: $\mathcal{X}_\varphi^\dagger \to S^\dagger$ is log smooth. By a similar argument as in ([Ols08] Lemma 3.1.11), the morphism is integral.

Consider the complement of the open torus in S, denoted by ∂S, and the complement of the open torus in $\mathcal{X}_\varphi$, denoted by $\partial\mathcal{X}$. We claim that the log structures constructed above are the divisorial log structures $M_{(S,\partial S)}$ and $M_{(\mathcal{X}_\varphi,\partial\mathcal{X})}$.

Proposition 2.19 (log structure)**.** *The log structures on $\mathcal{X}_\varphi^\dagger$ and $S^\dagger$ are the divisorial log structures from the toric boundaries.*

Proof. Both $S^\dagger$ and $\mathcal{X}_\varphi^\dagger$ are fine and saturated, and log smooth over $\operatorname{Spec} k^\dagger$. By([Kat94] Proposition 8.3), they are log regular (of toric singularities) as defined in loc. cit. Notice that $S\backslash\partial S$ and $\mathcal{X}_\varphi\backslash\partial\mathcal{X}$ are exactly the open subsets where the log structures are trivial. The statements follows from ([Kat94] Theorem 11.6). Notice that although the paper [Kat94] only discusses the log structure over small Zariski site, the theorems still hold for log structures over small étale sites.[1] □

[1]Another way to prove this is to use the log structure on Zariski sites induced from the log structures over étale sites. Then use theorems in [Kat94]. By ([Ogu06] Chap. II Corollary 1.2.11), the sheaves over étale sites are isomorphic.

2.1.3 The Standard Families

Following [Ols08], define $N_i = S(\sigma_i)$, and $N_{\mathscr{P}} := \varinjlim N_i$ (in the category of integral monoids). And notice that $\mathrm{Hom}(\varinjlim N_i, \mathbf{Z}) \cong \varprojlim(\mathrm{Hom}(N_i, \mathbf{Z}))$, $N_{\mathscr{P}}^{\mathrm{gp}}$ is the dual of $\varprojlim(\mathrm{Hom}(N_i, \mathbf{Z}))$. Also consider the direct limit in the category of sets $S(\underline{Q}) := \varinjlim N_i$. As a set, $S(\underline{Q}) = S(Q)$. $\varprojlim(\mathrm{Hom}(N_i, \mathbf{Z}))$ is the group of integral piecewise linear functions on $C(\underline{Q})$, and is denoted by $PL(\mathscr{P}, \mathbf{Z})$. Define the natural map $\tilde{\varphi}' : S(\underline{Q}) \to N_{\mathscr{P}}$ by the universal property of $S(\underline{Q})$. $\tilde{\varphi}'$ should be regarded as the universal $\mathscr{P}$-piecewise linear function because $N_{\mathscr{P}}^{\mathrm{gp}} = (PL(\mathscr{P}, \mathbf{Z}))^*$. However, we only need convex functions and we want to neglect affine functions. Define $H_{\mathscr{P}} \subset N_{\mathscr{P}}^{\mathrm{gp}}$ to be the submonoid generated by

$$\alpha * \beta := \tilde{\varphi}'(\alpha) + \tilde{\varphi}'(\beta) - \tilde{\varphi}'(\alpha + \beta), \forall \alpha, \beta \in S(Q).$$

Notice that $PA(\mathscr{P}, \mathbf{Z}) = PL(\mathscr{P}, \mathbf{Z})$. The evaluation is a natural map from $H_{\mathscr{P}}$ to the space $PA(\mathscr{P}, \mathbf{Z})^*$. This map is injective because everything is integral. The image is in $(PA(\mathscr{P}, \mathbf{Z})/Aff)^*$, because $H_{\mathscr{P}} \subset \mathrm{ann}(Aff)$. Moreover, $H_{\mathscr{P}}$ is sharp because $\mathscr{P}$ is regular.

Proposition 2.20. *Let $C(\mathscr{P}, \mathbf{Z})$ be the submonoid of the image of integral convex functions in $PA(\mathscr{P}, \mathbf{Z})/Aff$, then*

$$H_{\mathscr{P}}^{\mathrm{sat}} = C(\mathscr{P}, \mathbf{Z})^{\vee}$$

Proof. It is easy to see that $\alpha * \beta \in C(\mathscr{P}, \mathbf{Z})^{\vee}$. It follows that $H_{\mathscr{P}} \subset C(\mathscr{P}, \mathbf{Z})^{\vee}$. On the other hand, for any $\bar{\psi} \in PA(\mathscr{P}, \mathbf{Z})/Aff$. Pick any lift

$\psi \in PA(\mathscr{P}, \mathbf{Z})$. ψ is a $\mathscr{P}$-piecewise affine function over $\underline{Q}$. $\psi(\alpha * \beta) \geqslant 0$ for all α, β if and only if it is convex. Therefore $(H_{\mathscr{P}})^{\vee} = C(\mathscr{P}, \mathbf{Z})$. We claim that $H_{\mathscr{P}}^{\mathrm{gp}} = \mathrm{ann}(Aff) \cap N_{\mathscr{P}}^{\mathrm{gp}}$. Assume this, then, since $H_{\mathscr{P}}$ is sharp, $H_{\mathscr{P}}^{\mathrm{sat}} = ((H_{\mathscr{P}})^{\vee})^{\vee} = C(\mathscr{P}, \mathbf{Z})^{\vee}$.

Now we prove the claim. By ([Ols08] Lemma 3.1.7), $H_{\mathscr{P}}$ is equal to the image of $SC_1(\mathbb{X}_{\geqslant 0})/B_1$. By the description of $SC_1(\mathbb{X}_{\geqslant 0})/B_1$ in the proof of ([Ale02] Lemma 2.9.5), it is a semi-group generated by integral points in the kernel that defines all the linear relations in $\mathbb{X}_{\mathbf{R}}$. Therefore $H_{\mathscr{P}}^{\mathrm{gp}}$ is saturated in $\mathrm{ann}(Aff)$. $\square$

Define the bundle $\mathbb{P}_{\mathscr{P}} = Q \times H_{\mathscr{P}}^{\mathrm{gp}}$. Up to a global affine function, the piecewise affine function φ' is equal to a piecewise affine function $\varphi : Q \to H_{\mathscr{P}}^{\mathrm{gp}}$. (Because the bending parameters are all in $H_{\mathscr{P}}^{\mathrm{gp}}$. We can simply require φ to be zero on some top-dimensional cell.) By definition, φ is $H_{\mathscr{P}}$-convex and integral. Let $k = \mathbf{Z}$, and define $S(Q_{\varphi})$ and $\mathcal{X}_{\mathscr{P}} := \mathcal{X}_{\varphi}$ as before. This is called the standard family.

Define $S(Q) \rtimes H_{\mathscr{P}}^{\mathrm{sat}}$ to be the monoid with underlying set $S(Q) \times H_{\mathscr{P}}^{\mathrm{sat}}$ and the addition given by

$$(\alpha, p) + (\beta, q) := (\alpha + \beta, p + q + \alpha * \beta).$$

From the above discussion, $R_{\varphi} \cong k[S(Q) \rtimes H_{\mathscr{P}}^{\mathrm{sat}}]$ and

$$\mathcal{X}_{\mathscr{P}} \cong \operatorname{Proj} k[S(Q) \rtimes H_{\mathscr{P}}^{\mathrm{sat}}] \to \operatorname{Spec} k[H_{\mathscr{P}}^{\mathrm{sat}}].$$

Therefore, our standard family is the saturation of the standard family in [Ols08]. We can use the pull–back log structures from the standard family in [Ols08], and denote them by $(\mathcal{X}_{\mathscr{P}}, M_{\mathscr{P}})$ over $(S, M_S) = \mathrm{Spec}(H_{\mathscr{P}} \to k[H_{\mathscr{P}}^{\mathrm{sat}}])$. By (loc. cit. 3.1.12), the log morphism $(\mathcal{X}_{\mathscr{P}}, M_{\mathscr{P}}) \to (S, M_S)$ is integral and log smooth.

Let's find out the bending parameters $\{p(\varsigma)\}$.

Proposition 2.21. *For any codimension-1 wall ρ, the bending parameter $p_\rho \in H_{\mathscr{P}}^{\mathrm{sat}}$ is described as follows. Let ρ be the common face of maximal cells σ_i and σ_j. For any $\mathscr{P}$-piecewise affine function $\psi \in PA(\mathscr{P}, \mathbf{R})$, ψ_i and ψ_j are affine extensions of $\psi|_{\sigma_i}$ and $\psi|_{\sigma_j}$. Let $\omega \in \overline{X}$ be a point that maps to a minimal generator of $C(\overline{X})/C(L(\rho) \cap \overline{X}) \cong \mathbf{Z}$ and is on the same side as σ_i, then the bending parameter is defined by*

$$p_\rho(\psi) = \psi_i(\omega) - \psi_j(\omega). \tag{2.4}$$

Proof. $\forall \psi \in PA(\mathscr{P}, \mathbf{R})$, we have

$$\tilde{\varphi}(\alpha)(\psi) + \tilde{\varphi}(\beta)(\psi) - \tilde{\varphi}(\alpha+\beta)(\psi) = \psi(\alpha) + \psi(\beta) - \psi(\alpha+\beta),$$
$$(\psi \circ \tilde{\varphi} - \psi)(\alpha+\beta) = (\psi \circ \tilde{\varphi} - \psi)(\alpha) + (\psi \circ \tilde{\varphi} - \psi)(\beta) \quad \forall \alpha, \beta.$$

It means $\psi \circ \tilde{\varphi} - \psi$ is a global linear function on $S(Q)$. It follows that

$$(\tilde{\varphi}_i - \tilde{\varphi}_j)(\psi) = \psi_i - \psi_j.$$

Therefore,

$$p_\rho(\psi) = \psi_i(\omega) - \psi_j(\omega).$$

□

2.2 The Secondary Fan is the Mori Fan

2.2.1 The Secondary Fan

In this section, we recall the definition of the secondary fan from [GKZ94]. Fix $Q \subset \overline{X}_{\mathbf{R}}$ a lattice polytope of full dimension g. Denote $Q(\mathbf{Z})$ by I, and the cardinality $|I|$ by N. For any face $F \subset Q$, define the normal fan $N_F Q = \{v \in X^*_{\mathbf{R}} : \langle u, v\rangle \leqslant \langle u', v\rangle,\ \forall u \in F, u' \in Q\}$ following [Ful93].

For any function ψ on I, we can construct a convex function and a paving. Consider the following subset of $\overline{X}_{\mathbf{R}} \oplus \mathbf{R}$

$$\{(\omega, y) : y \geqslant \psi(\omega), \omega \in I, y \in \mathbf{R}\}$$

Let G_ψ be the convex hull of this subset. We take the lower bound of G_ψ. The projection of the faces gives a paving $\mathscr{P}$ of Q, whose vertices are all in I. Moreover, the lower bound is a graph of some piecewise affine convex function g_ψ.

For an arbitrary triangulation $\mathscr{T}$, we can also define a $\mathscr{T}$-piecewise affine function $g_{\psi,\mathscr{T}} : Q \to \mathbf{R}$. For each $\omega \in I$, which is also an element of $\mathscr{T}$, we have $g_{\psi,\mathscr{T}}(\omega) = \psi(\omega)$. The function $g_{\psi,\mathscr{T}}$ is obtained by affinely interpolating ψ inside each simplex of $\mathscr{T}$.

We regard each function ψ as an element in $\mathbf{R}^I$.

Definition 2.22. Let $\mathscr{T}$ be a triangulation of Q. We shall denote by $\widetilde{C}(\mathscr{T})$ the cone in $\mathbf{R}^I$ consisting of functions $\psi : I \to \mathbf{R}$ with the following two properties:

a) The function $g_{\psi,\mathscr{T}} : Q \to \mathbf{R}$ is convex.

b) For any $\omega \in I$ but not a vertex of any simplex from $\mathscr{T}$, we have $g_{\psi,\mathscr{T}} \leqslant \psi(\omega)$.

If $\mathscr{T}$ is a coherent triangulation, then the map $\mathbf{R}^I \to PA(\mathscr{T}, \mathbf{R})$ which sends ψ to $g_{\psi,\mathscr{T}}$ is a surjection, and the evaluation map $PA(\mathscr{T}, \mathbf{R}) \to \mathbf{R}^I$ is injective. Thus $\mathbf{R}^I = PA(\mathscr{T}, \mathbf{R}) \times \mathbf{R}^{I_\emptyset}$, where $I_\emptyset = I \backslash \mathscr{T}$. The projection from $\mathrm{pr} : \mathbf{R}^I \to \mathbf{R}^{I_\emptyset}$ is described as follows. For any $\omega \in I_\emptyset$, ω is in the interior of some simplex $\sigma \in \mathscr{T}$, and assume $\omega = \sum_i a_i \omega_i$, for $\{\omega_i\}$ the set of vertices of σ. The ω-coordinate of $\mathrm{pr}(\psi)$ is $\psi(\omega) - \sum_i a_i \psi(\omega)$. Denote the convex piecewise affine $\mathbf{R}$-valued functions by $CPA(\mathscr{T}, \mathbf{R})$, $\widetilde{C}(\mathscr{T}) = CPA(\mathscr{T}, \mathbf{R}) \times \mathbf{R}^{I_\emptyset}_{\geqslant 0}$. Since $N^{\mathrm{gp}}_{\mathscr{T},\mathbf{R}} = (PA(\mathscr{T}, \mathbf{R}))^*$, we also have $(\mathbf{R}^I)^* = N^{\mathrm{gp}}_{\mathscr{T},\mathbf{R}} \times (\mathbf{R}^{I_\emptyset})^*$. We need the following proposition ([GKZ94] Chap. 7, Prop. 1.5).

Proposition 2.23. *Fix Q and I. The cones $\widetilde{C}(\mathscr{T})$ for all the coherent triangulations of Q together with all faces of these cones form a complete generalized fan in $\mathbf{R}^I$.*

Remark 2.24. Notice that if g_ψ gives a paving $\mathscr{P}$ that is not a triangulation, then ψ is in the face of some $\widetilde{C}(\mathscr{T})$.

Now we construct the secondary polytope. Equip $\overline{X}_{\mathbf{R}}$ with the usual Lebesgue measure. Let $\mathscr{T}$ be a triangulation of Q. Define $\varphi_{\mathscr{T}} \in (R^I)^*$

$$\varphi_{\mathscr{T}}(\psi_\omega) := \sum_{\sigma:\omega \in \mathrm{Vert}(\sigma)} \mathrm{Vol}(\sigma).$$

Definition 2.25. The secondary polytope $P(Q)$ is the convex hull in the space $(\mathbf{R}^I)^*$ of the vectors $\varphi_{\mathscr{T}}$ for all the triangulations $\mathscr{T}$ of Q.

There is a standard basis in $\mathbf{R}^I$, the characteristic functions of $\omega \in I$, denoted by $\{\psi_\omega\}$. The dual basis is denoted by $\{e_\omega\}$. Consider the linear map p

$$p : (\mathbf{Z}^I)^* \to \mathbb{X},$$
$$e_\omega \mapsto (\omega, 1).$$

Denote $\ker p$ by $\mathbb{L}$. $\mathbb{L} = Aff^\perp$. The dual $\mathbb{L}^*$ is defined to be the quotient of $\mathbf{Z}^I$ by $Aff(\overline{X}, \mathbf{Z})$. It is not necessarily torsion free. Let the free part be $\mathbb{L}_f^*$. Define $q : \mathbf{Z}^I \to \mathbb{L}_f^*$. The kernel of q is the saturation of $Aff(\overline{X}, \mathbf{Z})$ in $\mathbf{Z}^I$.

Definition 2.26 (the secondary fan)**.** The image of $\widetilde{C}(\mathscr{T})$ with all their faces form a complete fan $\Sigma(Q)$ in $\mathbb{L}_{\mathbf{R}}^* = \mathbf{R}^I/Aff$, and is called the secondary fan. The image of $\widetilde{C}(\mathscr{T})$ in $\mathbb{L}_{\mathbf{R}}^*$ is denoted by $C(\mathscr{T})$.

Remark 2.27. This definition is different from that in [GKZ94]. We take the negative of GKZ's fan and then take the quotient.

Lemma 2.28. *We have*

$$C(\mathscr{T}) = C(\mathscr{T}, \mathbf{R}) \times \mathbf{R}_{\geqslant 0}^{I_\emptyset}, \quad C(\mathscr{T})^\vee = H_{\mathscr{T},\mathbf{R}}^{\text{sat}} \times \mathbf{R}_{\geqslant 0}^{I_\emptyset}.$$

Let $\triangle_g$ be the simplex generated by e_ω in $(\mathbf{R}^I)^*$. The linear map p is induced from the affine projection $p_a : \triangle_g \to Q$. The secondary polytope

$P(Q)$ is also defined to be the $(g+1)$-times dilated Minkowski integral of the fibers of p_a,

$$P(Q) := (g+1)\int_Q p_a^{-1}(x)dx.$$

Proposition 2.29. *The main properties of $P(Q)$ are*

a) The secondary polytope $P(Q)$ has dimension $N-g-1$.

b) Vertices of $P(Q)$ are precisely the characteristic functions $\varphi_{\mathscr{T}}$ for all coherent triangulations $\mathscr{T}$ of Q.

c) For any triangulation $\mathscr{T}$, the normal cone coincides with the cone $\widetilde{C}(\mathscr{T})$.

d) The affine span of $P(Q)$ is equal to

$$\overline{\mathbb{L}}_{\mathbf{R}} = \left\{\varphi \in (\mathbf{R}^I)^* : \sum_{\omega\in I}\psi_\omega(\varphi) = (g+1)\cdot Vol(Q),\ \sum_{\omega\in I}\psi_\omega(\varphi)\cdot\omega = (g+1)\int_Q xdx\right\}.$$

By Proposition 2.29, $\Sigma(Q)$ is the normal fan of the secondary polytope in the affine space $\overline{\mathbb{L}}_{\mathbf{R}}$.

2.2.2 The Mirror Symmetry Interpretation of the Secondary Fan

Let Y be the dual lattice of X. Given the lattice polytope $Q \subset \overline{X}_{\mathbf{R}}$, consider the normal fan Δ_Q. For any face $F \subset Q$, following [Ful93], define the normal cone $\sigma_F Q = \{v \in Y_{\mathbf{R}} : \langle u, v\rangle \leqslant \langle u', v\rangle,\ \forall u \in F, u' \in Q\}$. The collection of cones $\sigma_F Q$, as Q varies over the faces of Q, form a complete fan which also defines the toric variety $X_Q = X_{\Delta_Q}$. Let A denote the set of all primitive vectors v of the rays $\Delta_Q(1)$ in the fan Δ_Q, define the lattice polytope $P = \mathrm{Conv}\{v\}_{v\in A}$ in $Y_{\mathbf{R}}$.

Let T_X be the algebraic torus with the character group Y. Regard P as the Newton polytope of the Laurent polynomial

$$W(z) = -1 + \sum_{v\in A} z^v.$$

For a toric variety X_{Δ_Q}, the toric boundary divisor D is anticanonical. The mirror of the toric pair (X_{Δ_Q}, D) is defined to be the Landau–Ginzburg model (T_X, W) [Abo06]. The polytope Q gives rise to a 1-dimensional degeneration family of the Landau–Ginzburg models $(\check{\pi} : \overline{\mathfrak{Y}} \to \mathbf{A}^1, W_t)$ as follows: The vertices of the polytope Q defines a piecewise linear, strictly convex function $\check{\psi}$ over the fan Δ_Q. Define

$$W_t(z) = -1 + \sum_{v\in A} z^v t^{-\check{\psi}(v)}.$$

The strict convex piecewise linear function $\check{\psi}$ defines an integral paving $\check{\mathscr{Q}}$ of P. Take the discrete Legendre transform $(\overline{X}_{\mathbf{R}}, \mathscr{Q}, \psi)$ of the triple $(P, \check{\mathscr{Q}}, \check{\psi})$. Let Π be the non-smooth locus of ψ. As t goes to zero, after some reparametrization, the log amoeba of W_t converges to the tropical hypersurface Π of $\overline{X}_{\mathbf{R}}$ ([Mik04] Theorem 5). Notice that by definition of $\check{\psi}$, up to a negative sign, the Legendre transform of the origin inside P is the lattice polytope Q. Therefore, if we want to get Π as a tropical divisor in the limit, we need to construct a degeneration with Q as the dual intersection complex. Let $\overline{\Sigma}$ be the fan consisting of the faces of the rational polyhedral cone $C(Q) \subset \mathbb{X}_{\mathbf{R}}$. Let

$\overline{\mathcal{Y}}$ denote the affine toric variety $X_{\overline{\Sigma}}$. There is a natural morphism $\breve{\pi} : \overline{\mathcal{Y}} \to \mathbf{A}^1$ with the generic fiber T_X.

There is another interpretation of the mirror family $\overline{\mathcal{Y}} \to \mathbf{A}^1$. For each cone $\sigma \in \Delta_Q$, consider the cone of 1-higher dimension

$$\tilde{\sigma} = \{(u,t); u \in \sigma, t + \check{\psi}(u) \geqslant 0\} \subset \mathbb{Y}_{\mathbf{R}}.$$

Let $\widetilde{\Delta}_Q$ be the fan in $\mathbb{Y}_{\mathbf{R}}$ consisting of the cones $\tilde{\sigma}$ for $\sigma \in \Delta_Q$ and their faces. This is the fan for the total space of $\mathcal{L}$ ([CLS11] Proposition 7.3.1). The primitive vectors in the rays $\widetilde{\Delta}_Q(1)$ are $(v, -\check{\varphi}(v))$. Therefor, the mirror of the total space $\mathcal{L}$ is the torus of dimension $g+1$ $T_{\mathbb{X}}$ with the superpotential $W(t,z) := W_t(z)$. $\overline{\mathcal{Y}}$ is the partial compactification of $T_{\mathbb{X}}$ with the limit of W.

Let $\mathscr{P}$ be a paving of Q. For any cell $\sigma \in \mathscr{P}$, the cone $C(\sigma)$ is a strongly convex rational polyhedral cone in $\mathbb{X}_{\mathbf{R}}$. The cones $\{C(\sigma)\}$ form a fan denoted by $\Sigma_{\mathscr{P}}$. Let $\mathcal{Y}_{\mathscr{P}}$ denote the toric variety $X_{\Sigma_{\mathscr{P}}}$. The natural morphism $f : \mathcal{Y}_{\mathscr{P}} \to \overline{\mathcal{Y}}$ is birational and proper. By ([GD63] Theorem 3.2.1), $f_*\mathcal{O}_{\mathcal{Y}_{\mathscr{P}}}$ is a coherent $\mathcal{O}_{\overline{\mathcal{Y}}}$-module. Since $\overline{\mathcal{Y}}$ is normal and f is birational, $R_0 := \Gamma(\overline{\mathcal{Y}}, \mathcal{O}_{\overline{\mathcal{Y}}}) \cong \Gamma(\mathcal{Y}_{\mathscr{P}}, \mathcal{O}_{\mathcal{Y}_{\mathscr{P}}})$. If $\mathscr{P}$ is coherent, $\Sigma_{\mathscr{P}}$ has a strictly convex support function. By ([CLS11] Theorem 7.2.4), $f : \mathcal{Y}_{\mathscr{P}} \to \overline{\mathcal{Y}}$ is projective. Denote the set of rays in $\Sigma_{\mathscr{P}}$ by $\Sigma_{\mathscr{P}}(1)$. There is a canonical bijection $\Sigma_{\mathscr{P}}(1) \to I \cap \mathscr{P}$.

A projective morphism $f : \mathcal{Y} \to \overline{\mathcal{Y}}$ is a relative minimal model if $K_{\mathcal{Y}} + Y_0$ is f-nef, $\mathcal{Y}$ has terminal singularities and is $\mathbf{Q}$-factorial. In our case, since $\mathcal{Y}_{\mathscr{P}}$ is a toric variety, $K + Y_{\mathscr{P},0}$ is trivial. By ([CLS11] Exercise 8.2.14 (a)), $\mathcal{Y}_{\mathscr{P}}$ is Gorenstein. By ([CLS11] Proposition 11.4.12 (b)), $\mathcal{Y}_{\mathscr{P}}$ has canonical singularities, and if $I = Q(\mathbf{Z}) \subset \mathscr{P}$, $\mathcal{Y}_{\mathscr{P}}$ has terminal singularities. Moreover, $\mathcal{Y}_{\mathscr{P}}$ is an orbifold if and only if it is $\mathbf{Q}$-factorial, if and only if $\mathscr{P}$ is a triangulation ([CLS11] Proposition 4.2.7 & Theorem 11.4.8). Therefore, we have

Lemma 2.30. *A toric model $\mathcal{Y}_{\mathscr{P}} \to \overline{\mathcal{Y}}$ is a relative minimal model if $\mathscr{P}$ is a coherent triangulation, and $\Sigma_{\mathscr{P}}(1) = I$.*

If $\mathcal{Y}_{\mathscr{P}}$ is a relative minimal model, we simply denote $\Sigma_{\mathscr{P}}(1)$ by $\Sigma(1)$, since it doesn't depend on the decomposition. Any two relative minimal models are $\mathbf{Q}$-factorial, and isomorphic up to codimension 1. Therefore, they all have the same pseudo-effective cone, and the same Mori fan. Therefore we say that the Mori fan is canonical.

The definition of the Mori fan is from [HK00]. Since $\mathcal{Y}_{\mathscr{P}}$ is not projective, we have to modify the definition, and work in the relative case. Assume the morphism between schemes $X \to \overline{X}$ is projective, and $\overline{X} = \operatorname{Spec} R_0$ is affine.

Definition 2.31. For a divisor D on X, the section ring is the graded R_0-algebra

$$\mathrm{R}(X, D) := \oplus_{n \in \mathbf{N}} \mathrm{H}^0(X, \mathcal{O}(nD)).$$

If $\mathrm{R}(X, D)$ is finitely generated over R_0, and D is effective, then there is a rational map over R_0,

$$f_D : X \dashrightarrow \operatorname{Proj} \mathrm{R}(X, D).$$

Definition 2.32. Let D, and D' be two $\mathbf{Q}$-Cartier divisors on X with section rings finitely generated over R_0. Then we say D and D' are Mori equivalent if the rational maps f_D and $f_{D'}$ have the same Stein factorization.

Definition 2.33 (the Mori chamber)**.** Let $X \to \overline{X}$ be as above. Assume that $\mathrm{R}(X, D)$ is finitely generated over R_0 for all divisors D on X, and $\mathrm{Pic}(X)_{\mathbf{Q}} = \mathrm{NS}_{\mathbf{Q}}(X)$. By a Mori chamber of $\mathrm{NS}_{\mathbf{R}}(X)$, we mean the closure of an equivalence class whose interior is open in $\mathrm{NS}_{\mathbf{R}}(X)$.

Remark 2.34. Different Mori chambers are always disjoint.

Definition 2.35. If all the Mori chambers with their faces form a fan in $\mathrm{NS}_{\mathbf{R}}(X)$, it is called the Mori fan of X.

Fix an arbitrary paving $\mathscr{P}$ and consider the toric variety $\mathcal{Y}_{\mathscr{P}}$. $PA(\mathscr{P}, \mathbf{Z})$ is a sublattice of $\mathbf{Z}^I$ of finite index. It might not be saturated. Recall a Cartier divisor for the toric variety $\mathcal{Y}_{\mathscr{P}}$ is described in terms of an integral piecewise linear function ψ over the fan $\Sigma_{\mathscr{P}}$. Linear functions correspond to trivial Cartier divisors, and convex functions correspond to nef Cartier divisors. Therefore, $\overline{\mathcal{K}}(\mathcal{Y}_{\mathscr{P}}) = C(\mathscr{P}, \mathbf{R})$, and $\overline{\mathrm{NE}}(\mathcal{Y}_{\mathscr{P}}) = H^{\mathrm{sat}}_{\mathscr{P},\mathbf{R}}$.

If $I \subset \mathscr{P}$, the exact sequence for the Weil divisor class group is

$$0 \longrightarrow Aff(\overline{X}, \mathbf{Z}) \longrightarrow \mathbf{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(\mathcal{Y}_{\mathscr{P}}) \longrightarrow 0.$$

Recall $\mathbb{X} = Aff(\overline{X}, \mathbf{Z})^*$, we have

$$\mathbb{L}^* = \mathrm{Cl}(\mathcal{Y}_{\mathscr{P}}), \quad \mathbb{L} = \mathrm{Hom}(\mathrm{Cl}(\mathcal{Y}_{\mathscr{P}}), \mathbf{Z}).$$

For a $\mathbf{Q}$-factorial toric variety $\mathcal{Y}_{\mathscr{P}}$,

$$\mathrm{Cl}(\mathcal{Y}_{\mathscr{P}})_{\mathbf{Q}} \cong \mathrm{NS}_{\mathbf{Q}}(\mathcal{Y}_{\mathscr{P}}) \cong \mathrm{Pic}(\mathcal{Y}_{\mathscr{P}})_{\mathbf{Q}}.$$

Therefore if $\mathcal{Y}_{\mathscr{P}}$ is a relative minimal model, we can identify $\mathbb{L}^*$ with $\mathrm{NS}(\mathcal{Y}_{\mathscr{P}})$, and say that the Mori fan is fan supported in $\mathbb{L}^*_{\mathbf{R}}$. The Weil divisor corresponding to the primitive vector $(\omega, 1)$ is denoted by D_ω. For a $\mathbf{R}$-Weil divisor $D = \sum_{\omega\in\Sigma(1)} a_\omega D_\omega$, the associated function is defined by $\psi_D(\omega) = a_\omega$. The equivalent class in $\mathbb{L}^*_{\mathbf{R}}$ is also denoted by ψ_D.

Theorem 2.36. *If $\mathcal{Y}_{\mathscr{P}}$ is a relative minimal model, the Mori fan exists. It is the secondary fan $\Sigma(Q)$ under the identification above.*

Proof. We show that each full dimensional cone $C(\mathscr{T})$ in the secondary fan is a Mori chamber. Fix such a coherent triangulation $\mathscr{T}$. Recall $C(\mathscr{T}) = C(\mathscr{T}, \mathbf{R}) \times \mathbf{R}^{I_\emptyset}_{\geqslant 0}$. For any Cartier divisor D in the interior $C(\mathscr{T})^\circ$, let ψ_D be the associated function in $\mathbf{Z}^I$. $\psi_D = \psi_E + \psi_F$ for $\psi_E \in C(\mathscr{T}, \mathbf{Z})$ and $\psi_F \in \mathbf{Z}^{I_\emptyset}_{\geqslant 0}$. The corresponding divisors are E and F. We can multiply D by a positive integer to make both E and F Cartier.

Each Cartier divisor $D = \sum_{\omega\in\Sigma(1)} a_\omega D_\omega$ defines a polyhedron

$$P_D := \left\{ m \in Aff(\overline{X}_{\mathbf{R}}, \mathbf{R}) : \langle m, (\omega, 1)\rangle \geqslant -a_\omega, \forall \omega \in \Sigma(1) \right\}.$$

By ([CLS11] Proposition 4.3.3),

$$\Gamma(\mathcal{Y}_{\mathscr{P}}, \mathcal{O}_{\mathcal{Y}_{\mathscr{P}}}(D)) = \bigoplus_{m\in P_D\cap\mathbb{X}^*} k \cdot \mathrm{X}^m.$$

Notice that for any $\omega \in I_\emptyset$, $\omega \in \sigma$ for some simplex $\sigma \in \mathscr{P}$. Let the vertices of σ be $\{\omega_i\}$, $\psi_F \in \mathbf{R}_{\geqslant 0}^{I_\emptyset}$ means that the inequalities on $\{\omega_i\}$ imply the inequality on ω. Therefore $P_D = P_E$. Consider g_{ψ_E}, a $\mathscr{T}$-piecewise affine function. It is strict convex because D is in $C(\mathscr{T})^\circ$. Again multiply D by a positive integer to make g_{ψ_E} integral with respect to $\mathscr{T}$. The corresponding Cartier divisor E is ample for $\mathcal{Y}_{\mathscr{T}}$. Therefore

$$\operatorname{Proj} \mathrm{R}(\mathcal{Y}_{\mathscr{P}}, D) = \operatorname{Proj} k[S(P_D)] \cong \operatorname{Proj} \mathrm{R}(\mathcal{Y}_{\mathscr{T}}, E) \cong \mathcal{Y}_{\mathscr{T}}.$$

It follows that $f_D : \mathcal{Y}_{\mathscr{P}} \dashrightarrow \mathcal{Y}_{\mathscr{T}}$ is the rational map defined by $\Sigma_{\mathscr{P}} \to \overline{\Sigma} \leftarrow \Sigma_{\mathscr{T}}$ and $D = f_D^*(E) + F$ for E ample, and F f_D-exceptional. That means each $C(\mathscr{T})^\circ$ is contained in one Mori chamber, and different $C(\mathscr{T})^\circ$ are contained in different Mori chambers. Since $C(\mathscr{T})$ and their faces form a complete fan $\Sigma(Q)$ in $\mathbb{L}_{\mathbf{R}}^*$, the Mori chambers agree with $\{C(\mathscr{T})\}$. As a result, the Mori fan exists and is the same with $\Sigma(Q)$. $\square$

Corollary 2.37. All relative minimal models $\mathcal{Y} \to \overline{\mathcal{Y}}$ are toric models $\mathcal{Y}_{\mathscr{P}} \to \overline{\mathcal{Y}}$ for some minimal triangulation $\mathscr{P}$.

Corollary 2.38. If $\mathcal{Y}_{\mathscr{P}}$ is a relative minimal model, then

$$\overline{\operatorname{Eff}}(\mathcal{Y}_{\mathscr{P}}) = \operatorname{Eff}(\mathcal{Y}_{\mathscr{P}}) = \mathrm{NS}_{\mathbf{R}}(\mathcal{Y}_{\mathscr{P}}) = \operatorname{Pic}(\mathcal{Y}_{\mathscr{P}})_{\mathbf{R}}.$$

and the moving cone is

$$\operatorname{Mov}(\mathcal{Y}_{\mathscr{P}}) = \bigcup_{I \subset \mathscr{T}} C(\mathscr{T}).$$

Corollary 2.39. The union $\bigcup_{I\subset\mathscr{T}} \overline{C}(\mathscr{T})$ is a convex polyhedral cone.

Proof. This is because the closure of the moving cone is always convex. It is also easy to prove the statement directly. $\square$

We generalize the notion of Mori dream spaces to varieties that are projective over affine varieties. Corollary 2.38 implies that the relative minimal models are Mori dream spaces. According to [HK00], the Mori theory here is an instance of the theory of VGIT. In this case, $\mathcal{Y}_{\mathscr{P}}$ are all GIT quotients by Cox's construction. Define $G = \mathrm{Hom}(\mathrm{Cl}(\mathcal{Y}_{\mathscr{P}}), \mathbb{G}_m)$. The homomorphism $\mathbf{Z}^{\Sigma(1)} \to \mathrm{Cl}(\mathcal{Y}_{\mathscr{P}})$ gives a G-action on $\mathbf{A}^{\Sigma(1)}$. The problem is how to take the quotient of $\mathbf{A}^{\Sigma(1)}$ by G.

Since the story from this perspective is well presented in [CLS11], we relate our notations with those in [CLS11], for the convenience of the readers. The cone C_ν is identified with $C(Q)$, and we are in the situation of ([CLS11] Proposition 14.3.11). A stability condition is given by a character $\psi \in \widehat{G}_{\mathbf{R}} = \mathbb{L}^*_{\mathbf{R}} = C_\beta$. The secondary fan $\Sigma(Q)$ is a complete fan in C_β. Each maximal cone $C(\mathscr{T})$ is a GIT chamber. For any $\psi \in C(\mathscr{T})^\circ$, we have $\mathbf{A}^{\Sigma(1)}/\!/_\psi G \cong \mathcal{Y}_{\mathscr{T}}$. Therefore, we say that on the A-side, the secondary fan $\Sigma(Q)$ controls the variation of GIT quotients.

2.3 Gluing the Families

In this section, we go back to the B-side and construct the compactification $\mathscr{T}_Q$ of the moduli of toric pairs (X_Q, Θ).

Definition 2.40. A full dimensional simplex $\sigma \subset \overline{X}_{\mathbf{R}}$ is called regular if the set $\{(v_i, 1)\}$ form a basis of $\mathbb{X}$ for $\{v_i\}$ the set of vertices of σ. In other words, σ is regular if the cone $C(\sigma)$ is a regular polyhedral cone for the lattice $\mathbb{X}$ ([CLS11] Definition 1.2.16).

From now on, assume that Q contains a full dimensional regular simplex. We can always rescale Q so that it contains a standard cube in $\overline{X}_{\mathbf{R}}$. A standard cube contains a basis for X, and thus contains a regular simplex. For the toric variety X_Q, rescaling is only changing the polarization. Therefore, this hypothesis is mild. Under this assumption $\mathbb{L}^*$ is free, and the following exact sequence is exact on the right

$$0 \longrightarrow \mathbb{L} \longrightarrow (\mathbf{Z}^I)^* \xrightarrow{p} \mathbb{X} \longrightarrow 0. \tag{2.5}$$

Choose an arbitrary g-dimensional regular simplex $\sigma \subset Q$. Define a linear operator $L_\sigma : \mathbf{R}^I \to Aff$.

Definition 2.41. $L_\sigma(\psi)$ is the affine extension to $\overline{X}_{\mathbf{R}}$ of $\psi|_\sigma$, and then we restrict it to I .

Definition 2.42. $\Psi : I \to \mathbb{L}_{\mathbf{Q}}$ is defined by

$$\langle \overline{\psi}, \Psi(\omega) \rangle = \psi(\omega) - L_\sigma(\psi)(\omega), \ \forall \overline{\psi} \in \mathbb{L}^*_{\mathbf{Q}},$$

where $\overline{\psi}$ is the image of $\psi \in \mathbf{Q}^I$.

The definition of $\Psi(\omega)$ is independent of the choice of the representative ψ.

Lemma 2.43. *The values of Ψ are in $\mathbb{L}$.*

Proof. Since σ is regular, L_σ maps $\mathbf{Z}^I$ to $Aff(\overline{X}, \mathbf{Z})$. If $\overline{\psi}$ is in $\mathbb{L}^*$, $\langle \overline{\psi}, \Psi(\omega) \rangle$ is an integer. Hence $\Psi(\omega) \in \mathbb{L}$. $\square$

Let $\mathscr{T}$ be a coherent triangulation. Denote $\mathrm{Pic}(\mathcal{Y}_{\mathscr{T}}) \times \mathbf{Z}^{I_\emptyset}$ by $\mathbb{L}^*_{\mathscr{T}}$. We choose the integral structure on $\mathbb{L}_{\mathbf{R}}$ to be the dual $\mathbb{L}_{\mathscr{T}}$. $\mathbb{L}_{\mathscr{T}}$ contains $\mathbb{L}$, and depends on $\mathscr{T}$. Define $g_{\Psi,\mathscr{T}} : Q \to \mathbb{L}_{\mathbf{R}}$ by affinely interpolating Ψ inside each simplex of $\mathscr{T}$. By the mirror interpretation, we have $C(\mathscr{T}) = \overline{\mathcal{K}}(\mathcal{Y}_{\mathscr{T}}) \times \mathbf{R}^{I_\emptyset}_{\geqslant 0}$, $C(\mathscr{T})^\vee = \overline{\mathrm{NE}}(\mathcal{Y}_{\mathscr{T}}) \times (\mathbf{R}^{I_\emptyset}_{\geqslant 0})^* \subset \mathbb{L}_{\mathbf{R}}$, and $H^{gp}_{\mathscr{T}} = \mathrm{Pic}(\mathcal{Y}_{\mathscr{T}})^*$. As a result, $S_{C(\mathscr{T})} := C(\mathscr{T})^\vee \cap \mathbb{L}_{\mathscr{T}} = H^{\mathrm{sat}}_{\mathscr{T}} \times (\mathbf{N}^{I_\emptyset})^*$. If further $I \subset \mathscr{T}$, then $H^{\mathrm{sat}}_{\mathscr{T}} = S_{C(\mathscr{T})}$.

Lemma 2.44. *The $\mathscr{T}$-piecewise function $g_{\Psi,\mathscr{T}}$ is equal to the composition of the universal $\mathscr{T}$-piecewise function $\varphi : Q \to H^{\mathrm{gp}}_{\mathscr{T}}$ in Sect. 2.1.3 and the map $H^{\mathrm{gp}}_{\mathscr{T}} \to \mathbb{L}_{\mathscr{T}}$. For each codimension-1 wall ρ, the bending parameter p_ρ is the curve class of $V(C(\rho)) \in \mathrm{NE}(\mathcal{Y}_{\mathscr{T}})$. In particular, $g_{\Psi,\mathscr{T}}$ is integral with respect to $\mathbb{L}_{\mathscr{T}}$.*

Proof. For any codimension-1 wall ρ that is the intersection of the maximal cells $\sigma_i, \sigma_j \in \mathscr{T}$, we have, for $\psi \in \mathrm{Pic}(\mathcal{Y}_{\mathscr{T}})$,

$$g_{\Psi,\mathscr{T}}|_{\sigma_i}(\psi) - g_{\Psi,\mathscr{T}}|_{\sigma_j}(\psi) = \psi|_{\sigma_i} - \psi|_{\sigma_j}. \tag{2.6}$$

Compare with the bending parameters for φ in Equation (2.4), we see the bending parameters for $g_{\Psi_{\mathscr{T}}}$ are obtained from the bending parameters for φ via $H^{\mathrm{gp}}_{\mathscr{T}} \to \mathbb{L}_{\mathscr{T}}$. The interpretation of the bending parameters in terms of curve classes is a standard result in toric geometry ([CLS11] Proposition 6.3.8). $\square$

Remark 2.45. We make a base change from $\mathbb{L}$ to $\mathbb{L}_{\mathscr{T}}$ because we need the central fiber reduced.

Construct the polyhedron $Q_{g_{\Psi,\mathscr{T}}}$, the graded ring $R_{g_{\Psi,\mathscr{T}}} = k[S(Q_{g_{\Psi,\mathscr{T}}})]$, and the family $\mathcal{X}_{\mathscr{T}} := \operatorname{Proj} R_{g_{\Psi,\mathscr{T}}}$ over $U_{C(\mathscr{T})} := \operatorname{Spec} k[S_{C(\mathscr{T})}]$, with the morphism $\pi : \mathcal{X}_{\mathscr{T}} \to U_{C(\mathscr{T})}$. Take the π-ample line bundle $\mathcal{L} := \mathcal{O}(1)$. Define a section of $\mathcal{L}$

$$\vartheta := \sum_{\omega \in Q(\mathbf{Z})} X^{(\omega,\Psi(\omega))}.$$

By definition $(\omega, \Psi(\omega))$ is $C(\mathscr{T})^{\vee}$-above $(\omega, g_{\Psi,\mathscr{T}}(\omega))$, therefore $(\omega, \Psi(\omega)) \in Q_{g_{\Psi,\mathscr{T}}}(\mathbf{Z})$ and ϑ is a section of $\mathcal{L}$. Take the divisor $\Theta := (\vartheta)_0$. Moreover, $g_{\Psi,\mathscr{T}}(\omega) = \Psi(\omega)$ if and only if $\omega \in \mathscr{T} \cap I$. By Proposition 2.18, $(\mathcal{X}_{\mathscr{T}}, \mathcal{L}, \Theta, \varrho)$ is a stable toric pair over $U_{C(\mathscr{T})}$. By Cororllary 2.17, $\mathcal{X}_{\mathscr{T}}$ is the pull–back of the standard family for $\mathscr{T}$. Define the log structures on $\mathcal{X}_{\mathscr{T}}/U_{C(\mathscr{T})}$ to be the log structures pulled back from the standard family, and denote them by $(\mathcal{X}_{\mathscr{T}}, P_{\mathscr{T}})/(U_{C(\mathscr{T})}, M_{\mathscr{T}})$.

Lemma 2.46. *The family $\pi : (\mathcal{X}_{\mathscr{T}}, P_{\mathscr{T}}, \mathcal{L}, \vartheta, \varrho) \to (U_{C(\mathscr{T})}, M_{\mathscr{T}})$ is an object in $\mathscr{K}_Q(U_{C(\mathscr{T})})$, where $\mathscr{K}_Q$ is the stack defined in [Ols08].*

Proof. It suffices to check the definition of the stack $\mathscr{K}_Q$ in ([Ols08] 3.7.1). In particular, (vii) is satisfied because of (loc. cit. Lemma 3.1.24). $\square$

Define the chart $U = \coprod_{\mathscr{T}} U_{C(\mathscr{T})}$, where the index is over all coherent triangulation $\mathscr{T}$. By the 2-Yoneda lemma, We can pick a corresponding morphism $F : U \to \mathscr{K}_Q$. The next step is to define the pre-equivalence relation $R \to U \times U = \coprod_{\mathscr{T}_1, \mathscr{T}_2} U_{C(\mathscr{T}_1)} \times U_{C(\mathscr{T}_2)}$.

First, consider the case $\mathscr{T}_1 = \mathscr{T}_2$, and denote it by $\mathscr{T}$. Compare the two lattices $\mathbb{L}^*$ and $\mathbb{L}^*_{\mathscr{T}}$. The quotient $\mathbb{L}^*/\mathbb{L}^*_{\mathscr{T}}$ is a finite abelian group. Let $G_{\mathscr{T}}$ be the kernel of $T_{\mathbb{L}^*_{\mathscr{T}}} \to T_{\mathbb{L}^*}$, and $m_{\mathscr{T}}$ be the biggest order of elements in $\mathbb{L}^*/\mathbb{L}^*_{\mathscr{T}}$. Assume that $m_{\mathscr{T}}$ is invertible in k, then $G_{\mathscr{T}}$ is isomorphic to the constant group scheme with fiber $\mathbb{L}^*/\mathbb{L}^*_{\mathscr{T}}$, and is étale over k. Define $R_{\mathscr{T}} = U_{C(\mathscr{T})} \times G_{\mathscr{T}} \to U_{C(\mathscr{T})} \times U_{C(\mathscr{T})}$, where the first factor is the projection and the second factor is the group action. Denote the two projections from $R_{\mathscr{T}}$ to $U_{C(\mathscr{T})}$ by s and t. Since $G_{\mathscr{T}}$ is étale, s is the first projection $U_{C(\mathscr{T})} \times G_{\mathscr{T}}$ to $U_{C(\mathscr{T})}$, s is étale and surjective. The morphism t is the action $U_{C(\mathscr{T})} \times G_{\mathscr{T}} \to U_{C(\mathscr{T})}$. It is the composition of the projection and the isomorphism $(x, g) \to (gx, g)$, and is also étale and surjective. We get an étale pre-equivalence relation $R_{\mathscr{T}}$. The coarse moduli space of the stack $[U_{C(\mathscr{T})}/R_{\mathscr{T}}] = [U_{C(\mathscr{T})}/G_{\mathscr{T}}]$ is the toric variety defined by the cone $C(\mathscr{T})^\vee$ and lattice $\mathbb{L}$.

Proposition 2.47. *We have a natural injective morphism $R_{\mathscr{T}} \to U_{C(\mathscr{T})} \times_{\mathscr{K}_Q} U_{C(\mathscr{T})}$.*

Proof. We claim that each element in $G_{\mathscr{T}}$ induces an isomorphism in $\mathscr{K}_Q(U_{C(\mathscr{T})})$. Let ζ be an S-point of $G_{\mathscr{T}}$ for some k-scheme S. The element ζ is acting on the algebra $R_S := \mathcal{O}_S[S_{C(\mathscr{T})}]$ by

$$g : \mathrm{X}^p \mapsto \mathrm{X}^p(\zeta)\mathrm{X}^p.$$

The pull–back $(g^*\mathcal{X}_{\mathscr{T}}, g^*\mathcal{L})$ is $\operatorname{Proj} R_S \otimes_{R_S} R_{g_{\Psi,\mathscr{T}}}$, where the map $R_S \to R_S$ is g. We use $R_{g_{\Psi,\mathscr{T}}} = R_S[S(Q_{g_{\Psi,\mathscr{T}}})] \subset R_S[\mathbb{X} \times \mathbb{L}_{\mathscr{T}}]$. Then $(g^*\mathcal{X}_{\mathscr{T}}, g^*\mathcal{L})$ is isomorphic to $(\mathcal{X}_{\mathscr{T}}, \mathcal{L})$ over $U_{C(\mathscr{T})}$ by an isomorphism

$$\begin{aligned} F_\zeta : g^* R_{g_{\Psi,\mathscr{T}}} = R_S \otimes_{R_S} R_{g_{\Psi,\mathscr{T}}} &\longrightarrow R_{g_{\Psi,\mathscr{T}}}, \\ 1 \otimes \mathrm{X}^{(n,\alpha,p)} &\longmapsto \mathrm{X}^p(\zeta)\mathrm{X}^{(n,\alpha,p)}. \end{aligned}$$

The isomorphism F_ζ preserves the $\mathbb{X}$-grading. Therefore $G_{\mathscr{T}}$ is acting on $(\mathcal{X}_{\mathscr{T}}, \mathcal{L})$ commuting with π and the action ϱ. The action on the log structures are the modifications by $H_{\mathscr{T}} \to \mathcal{O}_S^*$ and $S(Q) \rtimes H_{\mathscr{T}} \to \mathcal{O}_S^*$

$$p \mapsto 1, \tag{2.7}$$

$$(\alpha, p) \mapsto \mathrm{X}^{g_{\Psi,\mathscr{T}}(\alpha)}(\zeta). \tag{2.8}$$

The induced log structure is isomorphic to the original log structure, and thus $G_{\mathscr{T}}$ preserves the log morphism. The action by $G_{\mathscr{T}}$ also preserves the section ϑ because $\Psi(\omega) \in \mathbb{L}$ for all $\omega \in Q(\mathbf{Z})$.

The action of $G_{\mathscr{T}}$ exchanges with any base change $S \to U_{C(\mathscr{T})}$, and is faithful. $\square$

Next, consider two different Mori chambers $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$. Assume that $\tau := C(\mathscr{T}_1) \cap C(\mathscr{T}_2)$ is a wall. We want to glue the two associated families together. Regard $\omega \in I$ as a vector in $\mathbb{X}_{\mathbf{R}}$. According to ([CLS11] Chapter 15.3), there are only two cases.

The first case corresponds to divisorial contraction, and is called the divisorial case. In this case, according to ([CLS11] Theorem 15.3.6), one of the triangulation, say $\mathscr{T}_1$, has more vertices than the other. Moreover, there exists $\omega \in I$ such that $\mathscr{T}_1$ is star subdivision of $\mathscr{T}_2$ at ω. Let σ_0 be the simplex which contains ω in $\mathscr{T}_2$. Assume the vertices of σ_0 are $\{v_0, \ldots, v_n\}$. For any $\sigma_i \in \mathscr{T}_1$ such that $\omega \in \overline{\sigma}_i$, define ψ_i to be the affine function over $\overline{\sigma}_i$ which is 1 at ω, and 0 at other vertices. Then

$$\begin{aligned} g^{12} &:= g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2} \\ &= \begin{cases} 0 & \text{if } x \notin \mathrm{Star}(\sigma_0) \\ \psi_i(x) q_\tau & \text{if } x \in \sigma_i \end{cases}, \end{aligned}$$

for some $q_\tau \in \mathbb{L}$.

Write ω as an affine combination of $\{v_0, \ldots, v_n\}$,

$$\omega = \sum_{i=0}^{n} a_i v_i, \quad \text{with } \sum_{i=0}^{g} a_i = 1.$$

Compute q_τ,

$$\begin{aligned} q_\tau &= g^{12}(\omega) \\ &= g_{\Psi,\mathscr{T}_1}(\omega) - g_{\Psi,\mathscr{T}_2}(\omega) \\ &= \Psi(\omega) - \sum_{i=0}^{g} a_i \Psi(v_i). \end{aligned}$$

Therefore, for any $\psi' \in \mathbf{R}^I$,

$$\begin{aligned} q_\tau(\psi') &= (\psi', \Psi(\omega)) - \sum_{i=0}^{n} a_i(\psi', \Psi(v_i)) \\ &= \psi'(\omega) - L_\sigma(\psi')(\omega) - \sum_{i=0}^{n} a_i \psi'(v_i) + \sum_{i=0}^{n} a_i L_\sigma(\psi')(v_i) \\ &= \psi'(\omega) - \sum_{i=0}^{n} a_i \psi'(v_i). \end{aligned}$$

It follows that $-q_\tau \in C(\mathscr{T}_1)^\vee$, and $q_\tau \in C(\mathscr{T}_2)^\vee$. Since $C_\tau^\vee = C(\mathscr{T}_1)^\vee + C(\mathscr{T}_2)^\vee$, $q_\tau \in (S_\tau^*)_{\mathbf{Q}}$.

Remark 2.48. Since $\mathscr{T}_1$, $\mathscr{T}_2$ give different integral structures on $\mathbb{L}_{\mathbf{R}}$, we haven't defined S_τ yet. However, $(S_\tau^*)_{\mathbf{Q}}$ is well defined.

The second case corresponds to a flip $\mathcal{Y}_{\mathscr{T}_1} \dashrightarrow \mathcal{Y}_{\mathscr{T}_2}$, and is called the flipping case. Since $\mathcal{Y}_{\mathscr{T}_1}$ and $\mathcal{Y}_{\mathscr{T}}$ are isomorphic up to codimension 1, $\mathscr{T}_1$ and $\mathscr{T}_2$ have the same $I_\emptyset$. Assume τ corresponds to the paving $\mathscr{P}$. In this case, the wall τ comes from a wall between $\overline{\mathcal{K}}(\mathcal{Y}_{\mathscr{T}_1})$ and $\overline{\mathcal{K}}(\mathcal{Y}_{\mathscr{T}_2})$, and thus corresponds to an extremal ray $\mathcal{R} \subset \overline{\mathrm{NE}}(\mathcal{Y}_{\mathscr{T}_1})$ or $-\mathcal{R} \subset \overline{\mathrm{NE}}(\mathcal{Y}_{\mathscr{T}_2})$. This curve class defines two sets

$$J_- := \{v \in I : D_v \cdot \mathcal{R} < 0\}, \quad J_+ := \{v \in I : D_v \cdot \mathcal{R} > 0\}.$$

Also for any $J \subset I \backslash I_\emptyset$, set

$$\sigma_J := \mathrm{Cone}(\{v : v \in J\}) \subset \mathbb{X}_{\mathbf{R}}.$$

Here are the facts we need from ([CLS11] Theorem 15.3.13).

a) Both J_+ and J_- have at least 2 elements.

b) Vectors in J_- and J_+ form an oriented circuit. That means there is one linear relationship
$$\sum_{i\in J_-} b_i v_i + \sum_{i\in J_+} b_i v_i = 0,$$
where $b_i > 0$ if $i \in J_+$, $b_i < 0$ if $i \in J_-$. And every proper subset is linearly independent. We normalize it so that $\sum_{i\in J_+} b_i = -\sum_{i\in J_-} b_i = 1$.

c) $\sigma_{J_-} \in \Sigma_{\mathscr{T}_1}$, $\sigma_{J_+} \in \Sigma_{\mathscr{T}_2}$, and $\sigma_{J_-\cup J_+} \in \Sigma_{\mathscr{P}}$.

d) All the non-simplicial cones of $\Sigma_{\mathscr{P}}$ are contained in $\mathrm{Star}(\sigma_{J_-\cup J_+})$. And
$$\Sigma_{\mathscr{T}_1}\backslash\mathrm{Star}(\sigma_{J_-}) = \Sigma_{\mathscr{P}}\backslash\mathrm{Star}(\sigma_{J_-\cup J_+}) = \Sigma_{\mathscr{T}_2}\backslash\mathrm{Star}(\sigma_{J_+}).$$

e) For any maximal non-simplicial cell $\sigma_\alpha \in \Sigma_{\mathscr{P}}$, there is a set $J_\alpha \subset I\backslash I_\emptyset$, such that $J_\alpha \cup J_- \cup J_+$ is the set of vertices of σ_α, and $|J_\alpha \cup J_- \cup J_+| = g+2$. Denote $\Sigma_{\mathscr{T}_1}|_{\sigma_\alpha}$ by Σ_-, and $\Sigma_{\mathscr{T}_2}|_{\sigma_\alpha}$ by Σ_+, then we have
$$\Sigma_- = \{\sigma_J : J \subset J_\alpha \cup J_+ \cup J_-, J_+ \not\subseteq J\}, \quad \Sigma_+ = \{\sigma_J : J \subset J_\alpha \cup J_+ \cup J_-, J_- \not\subseteq J\}.$$

Set
$$\omega = \sum_{i\in J_+} b_i v_i = \sum_{i\in J_-} -b_i v_i.$$

For each maximal non-simplicial cell σ_α, let $J^\alpha = J_\alpha \cup J_- \cup J_+$. For any pair $i \in J_-, j \in J_+$, Define $\sigma_{ij} = \sigma_{J^\alpha\backslash\{i\}} \cap \sigma_{J^\alpha\backslash\{j\}}$. Since vectors in J_+ and J_- form a circuit
$$\sigma_{ij} = \mathrm{Cone}(\{\omega, v : v \in J^\alpha\backslash\{i,j\}\})$$

and is a simplex.

For any such σ_{ij}, define ψ_{ij} to be the affine function over $\overline{\sigma_{ij}}$ that is 1 on ω and is 0 on other vertices. Consider

$$
\begin{aligned}
g^{12} &:= g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2} \\
&= \begin{cases} 0 & \text{if } x \in |\Sigma_{\mathscr{P}} \backslash \mathrm{Star}(\sigma_{J_- \cup J_+})| \\ \psi_{ij}(x) q_\tau & \text{if } x \in \sigma_{ij} \end{cases},
\end{aligned}
$$

for some $q_\tau \in \mathbb{L}$.

Compute q_τ,

$$
\begin{aligned}
q_\tau &= g^{12}(\omega) \\
&= g_{\Psi,\mathscr{T}_1}(\omega) - g_{\Psi,\mathscr{T}_2}(\omega) \\
&= \sum_{i \in J_-} (-b_i)\Psi(v_i) - \sum_{i \in J_+} b_i \Psi(v_i)
\end{aligned}
$$

Therefore, for any $\psi' \in \mathbf{R}^I$,

$$
q_\tau(\psi') = \sum_{i \in J_-} (-b_i)\psi'(v_i) - \sum_{i \in J_+} b_i \psi'(v_i) \tag{2.9}
$$

Since J_+ has at least two elements, pick v_k, v_l from J_+. $\sigma_k := \sigma_{J \backslash \{k\}}$ and $\sigma_l := \sigma_{J \backslash \{l\}}$ are two maximal cones in Σ_-. $\varsigma := \sigma_{J \backslash \{k,l\}}$ is a wall between them. By ([CLS11] Proposition 6.4.4.)

$$
V(C(\varsigma))(\psi') = \frac{\mathrm{mult}(\varsigma)}{\mathrm{mult}(\sigma_k)(-b_k)} \Big(\sum_{i \in J_-} (-b_i)\psi'(v_i) - \sum_{i \in J_+} b_i \psi'(v_i) \Big)
$$

Therefore

$$
q_\tau = \frac{\mathrm{mult}(\sigma_k)(-b_k)}{\mathrm{mult}(\varsigma)} [V(C(\varsigma))] = \frac{\mathrm{mult}(\sigma_k)(-b_k)}{\mathrm{mult}(\varsigma)} p_\varsigma.
$$

Since $[V(C(\varsigma))]$ is a curve class in $\mathcal{R}$, q_τ is a curve class in $\mathcal{R}$. It follows that $q_\tau \in C(\mathscr{T}_1)^\vee$, and $-q_\tau \in C(\mathscr{T}_2)^\vee$. Again $q_\tau \in (S_\tau^*)_{\mathbf{Q}}$.

Proposition 2.49. *If $\mathscr{T}_1$ and $\mathscr{T}_2$ are coherent triangulations such that $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$ are of maximal dimension, and $\tau = C(\mathscr{T}_1) \cap C(\mathscr{T}_2)$ is a codimension-1 wall. Then $g^{12} = g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2}$ takes values in $(S_\tau^*)_{\mathbf{Q}}$.*

Corollary 2.50. If $\mathscr{T}_1$ and $\mathscr{T}_2$ are coherent triangulations such that $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$ are of maximal dimension. Let $\tau = C(\mathscr{T}_1) \cap C(\mathscr{T}_2)$ be the common face. Then $g^{12} = g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2}$ takes values in $(S_\tau^*)_{\mathbf{Q}}$.

Proof. First we claim that $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$ can be connected by a series of adjacent maximal cones $\{C_i\}_{0 \leqslant i \leqslant l}$ such that $C_0 = C(\mathscr{T}_1)$, $C_l = C(\mathscr{T}_2)$, and $\tau \subset C_i$ for all i. For the proof, look at the secondary polytope $P(Q) \subset \mathbb{L}_{\mathbf{R}}$. τ corresponds to a face F_τ of X which is itself a polytope. $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$ correspond to two vertices v_1 and v_2 of F_τ. And they can be connected by edges of F_τ. The vertices on these edges correspond to C_i we are seeking. This proves the claim.

Define $g^{i,i-1}$ for C_{i-1} and C_i as in the proposition 2.49. Then $g^{12} = g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2} = \sum_{i=1}^{l} g^{i,i-1}$, and $(\sum_{i=0}^{l} S_{C_i})_{\mathbf{Q}} \subset (S_\tau)_{\mathbf{Q}} = (S_{C_0} + S_{C_l})_{\mathbf{Q}} \subset (\sum_{i=0}^{l} S_{C_i})_{\mathbf{Q}}$. It follows that g^{12} takes values in $(S_\tau^*)_{\mathbf{Q}}$. $\square$

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two coherent triangulations such that $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$ are maximal cones. Let $\tau = C(\mathscr{T}_1) \cap C(\mathscr{T}_2)$. Define $\mathbb{L}_\tau$ to be the lattice generated by $S_{C(\mathscr{T}_1)}^{\mathrm{gp}}$ and $S_{C(\mathscr{T}_2)}^{\mathrm{gp}}$. Since $S_{C(\mathscr{T}_1)}^{\mathrm{gp}}$ and $S_{C(\mathscr{T}_2)}^{\mathrm{gp}}$ are commensurable,

$\mathbb{L}_\tau$ is commensurable to both of them. Define $S_\tau = (C(\mathscr{T}_1)^\vee + C(\mathscr{T}_2)^\vee) \cap \mathbb{L}_\tau$. Let $U_\tau := \operatorname{Spec} \mathbf{Z}[S_\tau]$. The inclusions of monoids $S_{C(\mathscr{T}_i)} \to \tau^\vee \cap S^{\mathrm{gp}}_{C(\mathscr{T}_i)} \to S_\tau$ define morphisms $p_{\tau,\mathscr{T}_i} : U_\tau \to U_{C(\mathscr{T}_i)}$.

Lemma 2.51. *The morphisms $p_{\tau,\mathscr{T}_i} : U_\tau \to U_{C(\mathscr{T}_i)}$ are both étale for $i = 1, 2$.*

Proof. Denote the lattice $\mathbb{L}_{\mathscr{T}_i}$ by $\mathbb{L}_i$. This is the lattice used to define $S_{C(\mathscr{T}_i)}$. We only need to show that the morphism $U_\tau \to \operatorname{Spec} k[\tau^\vee \cap \mathbb{L}_i]$ is étale. The cone τ is associated to the paving $\mathscr{P}$. Let N_τ be the lattice $PA(\mathscr{P}, \mathbf{Z})/Aff = \operatorname{Pic}(\mathfrak{Y}_{\mathscr{P}})$. We claim that $N_\tau = N_{\tau,\mathbf{R}} \cap \mathbb{L}_i^*$ for $i = 1, 2$. Notice that $\mathscr{P}$ is coarser than both $\mathscr{T}_i$. Assume that $\{\sigma_{jk}\}$ is a collection of top-dimensional cells in $\mathscr{T}_i$, and for each k, the union of the closure of σ_{jk} is the closure of a top-dimensional cell σ_k in $\mathscr{P}$. Then a piecewise affine function ψ in N_τ just means it is affine on each σ_k and are integral on each of top-dimensional cells. Therefore it is integral on each σ_{jk}, and it is in $N_{\tau,\mathbf{R}} \cap \mathbb{L}_i^*$. On the other hand, if $\psi \in N_{\tau,\mathbf{R}} \cap \mathbb{L}_i^*$, then it is affine on each σ_k and integral on each σ_{jk}. Since σ_{jk} is of top dimension, ψ is integral on σ_k by our definition of integrality. The claim is proved. Let I be the subset of vertices that are not in either of $\mathscr{T}_i$, i.e. $I := I^1_\emptyset \cap I^2_\emptyset$. Recall the cone $\tau = C(\mathscr{P}, \mathbf{R}) \times \mathbf{R}^I_{\geqslant 0} \times \{0\}$. As a result, we have the exact sequence,

$$0 \longrightarrow (S_\tau^*)_{\mathbf{Q}} \cap S^{\mathrm{gp}}_{C(\mathscr{T}_i)} \longrightarrow S^{\mathrm{gp}}_{C(\mathscr{T}_i)} \longrightarrow H^{\mathrm{gp}}_{\mathscr{P}} \times (\mathbf{Z}^I)^* \longrightarrow 0.$$

The image of the quotient $H^{\mathrm{gp}}_{\mathscr{P}} \times (\mathbf{Z}^I)^*$ is independent of i. Intersecting

$\tau^\vee$, we have

$$\begin{tikzcd}
0 \arrow[r] & (S_\tau^*)_{\mathbf{Q}} \cap \mathbb{L}_i \arrow[r] \arrow[d, "f^\flat"'] & \mathbb{L}_i \cap \tau^\vee \arrow[r] \arrow[d] & H_{\mathscr{P}}^{\mathrm{sat}} \times (\mathbf{N}^I)^* \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & (S_\tau^*)_{\mathbf{Q}} \cap \mathbb{L}_\tau \arrow[r] & S_\tau \arrow[r] & H_{\mathscr{P}}^{\mathrm{sat}} \times (\mathbf{N}^I)^* \arrow[r] & 0
\end{tikzcd}, \tag{2.10}$$

where $f^\flat$ is an injection with image of finite index. By ([Ogu06] Chap.I Proposition 1.1.4 part 2), the first square is a push–out diagram. Therefore, we get a pul–back diagram

$$\begin{tikzcd}
U_\tau \arrow[r, "g'"] \arrow[d, "p_{\tau,\mathscr{T}_i}"'] & \operatorname{Spec} k[(S_\tau^*)_{\mathbf{Q}} \cap \mathbb{L}_i] \arrow[d, "f"] \\
\operatorname{Spec} k[\tau^\vee \cap \mathbb{L}_i] \arrow[r, "g"] & \operatorname{Spec} k[(S_\tau^*)_{\mathbf{Q}} \cap \mathbb{L}_\tau],
\end{tikzcd}$$

where g and g' are fibrations with fibers $\operatorname{Spec} k[H_{\mathscr{P}}^{\mathrm{sat}} \times (\mathbf{N}^I)^*]$. Since f is étale, $p_{\tau,\mathscr{T}_i}$ is étale. □

Corollary 2.52. Let F_i be the face of $H_{\mathscr{T}_i}^{\mathrm{sat}} \cap \tau^\perp$, and $H_{\mathscr{T}_i,F_i}^{\mathrm{sat}}$ be the localization with respect to F_i. The following diagram commutes .

$$\begin{tikzcd}
C(\mathscr{T}_i)^\vee \cap \mathbb{L}_i \arrow[r] & \tau^\vee \cap \mathbb{L}_\tau \\
H_{\mathscr{T}_i}^{\mathrm{sat}} \arrow[r] \arrow[u] & H_{\mathscr{P}}^{\mathrm{sat}} \oplus F_i^{\mathrm{gp}}, \arrow[u]
\end{tikzcd}$$

where the bottom line is the localization.

Proof. It is implied by the diagram (2.10) in the above proof. We use the notations in the above proof. Let $I_i = I_\emptyset^i \backslash I$. Recall $C(\mathscr{T}_i)^\vee \cap \mathbb{L}_i = H_{\mathscr{T}_i}^{\mathrm{sat}} \times (\mathbf{N}^{I_\emptyset^i})^*$. The localization of $C(\mathscr{T}_i)^\vee \cap \mathbb{L}_i$ with respect to the face $\tau^\perp \cap \mathbb{L}_i$ is

$H^{\text{sat}}_{\mathscr{T}_i,F_i} \times (\mathbf{Z}^{I_i})^* \times (\mathbf{N}^I)^*$. Therefore, the top exact sequence in (2.10) shows that $H^{\text{sat}}_{\mathscr{P}}$ is the quotient of $H^{\text{sat}}_{\mathscr{T}_i,F_i}$ by the face F_i^{gp}. Since all the monoids are toric, we can choose splittings of the exact sequences. It is the commutative diagram we want. $\square$

Denote the group scheme in the étale pre-equivalence relation for $U_{C(\mathscr{T}_i)}$ by G_i, and the group action by $\rho_i : G_i \times U_{C(\mathscr{T}_i)} \to U_{C(\mathscr{T}_i)}$. Define $R_{ij} := G_i \times U_\tau \times G_j$ for $i \neq j$. Define $G_i \times U_\tau \to U_{C(\mathscr{T}_i)}$ by $\rho_i \circ (\mathrm{Id}, p_{\tau,\mathscr{T}_i})$. Then composed with the projection $G_i \times U_\tau \times G_j \to G_i \times U_\tau$, we get a morphism $s : R_{ij} \to U_{C(\mathscr{T}_i)}$. Define $t : R_{ij} \to U_{C(\mathscr{T}_j)}$ similarly. By Lemma 2.51, both s and t are étale.

Denote the toric monoid $S_{C(\mathscr{T}_i)}$ by P_i for $i = 1, 2$. We have defined $k[P_i]$-algebras $R_i := R_{g_{\Psi,\mathscr{T}_i}}$. The pull–back along $p_{\tau,\mathscr{T}_i}$ is $R_i|_{U_\tau} = k[S_\tau] \otimes_{k[P_i]} R_i$ is

$$k[S_\tau] \otimes_{k[P_i]} R_i \cong k[S(Q) \rtimes_i S_\tau].$$

The addition of $S(Q) \rtimes_i S_\tau$ is defined as

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + ng_{\Psi,\mathscr{T}_i}(\alpha) + mg_{\Psi,\mathscr{T}_i}(\beta) - (n+m)g_{\Psi,\mathscr{T}_i}(\gamma)).$$

We use the subscript $\rtimes_i$ to distinguish different additions on the same underlying set.

Consider $g^{12} = g_{\Psi,\mathscr{T}_1} - g_{\Psi,\mathscr{T}_2}$. Since $g_{\Psi,\mathscr{T}_i}$ are integral with respect to each integral structure, and $\mathbb{L}_\tau$ is a refinement of both the integral structure,

g^{12} is integral for $\mathbb{L}_\tau$. Therefore we can define a map

$$\begin{aligned} S(Q) \rtimes_1 S_\tau &\longrightarrow S(Q) \rtimes_2 S_\tau \\ (\alpha, p) &\longmapsto (\alpha, p + \deg(\alpha) g^{12}(\alpha)). \end{aligned}$$

It is a morphism between monoids and preserves the action of S_τ. Moreover, by Corollary 2.50, this is an isomorphism. Therefore, it induces an isomorphism between graded $k[S_\tau]$-algebras

$$\varphi_{\mathscr{T}_1 \mathscr{T}_2} : k[S(Q) \rtimes_1 S_\tau] \to k[S(Q) \rtimes_2 S_\tau].$$

Denote the pull back of family $\mathcal{X}_{\mathscr{T}_i}$ to U_τ by $\mathcal{X}_i$. It is defined as $\operatorname{Proj} k[S(Q) \rtimes_i S_\tau]$. The isomorphism $\varphi_{\mathscr{T}_1 \mathscr{T}_2}$ induces an isomorphism between $\mathcal{X}_i$'s.

The section ϑ for $U_{C(\mathscr{T})}$ is defined by $\sum_{\omega \in I} \mathrm{X}^{\Psi(\omega) - g_{\Psi, \mathscr{T}}(\omega)} \vartheta_\omega$, and is preserved by $\varphi_{\mathscr{T}_1 \mathscr{T}_2}$. Furthermore, the $\mathbb{X}$-grading is preserved. In other words, the line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, the sections Θ_1, Θ_2, the $\mathbb{T}$-action ϱ_1, ϱ_2 are all compatible under the isomorphism $\varphi_{\mathscr{T}_1 \mathscr{T}_2}$. The only thing left is the log structure.

Proposition 2.53. *The log structures on $U_{C(\mathscr{T}_i)}$ agree on U_τ, and has a chart $H_{\mathscr{P}}$. The log structures induced from P_i on $\mathcal{X}_i$ agree by the isomorphism $\varphi_{\mathscr{T}_1 \mathscr{T}_2}$, and has a chart $S(Q) \rtimes H_{\mathscr{P}}$. For any geometric point $\overline{s}$ in the interior of the closed orbit corresponding to τ, the above charts are good at $\overline{s}$.*

Proof. By Corollary 2.52, the map $H_{\mathscr{T}_i} \to H_{\mathscr{T}_i}^{\mathrm{sat}} \to H_{\mathscr{P}}^{\mathrm{sat}} \oplus F_i^{\mathrm{gp}}$ is a chart for the log structure $M_{U_{C(\mathscr{T}_i)}}$ on U_τ. Choose a geometric point $\overline{s}$ in the interior of the closed orbit corresponding to τ. It induces a geometric point of $U_{C(\mathscr{T}_i)}$, still denoted by $\overline{s}$, by the étale map $U_\tau \to U_{C(\mathscr{T}_i)}$. Let the residue field of $\overline{s}$ be $k(\overline{s})$. Therefore, we get a map $H^{\mathrm{sat}}_{\mathscr{T}_i} \to k(\overline{s})$ such that F_i is the inverse image of $k(\overline{s})^*$. The quotient $H^{\mathrm{sat}}_{\mathscr{T}_i}/F_i = H^{\mathrm{sat}}_{\mathscr{P}}$ by Corollary 2.52. It induces a morphism $f : H_{\mathscr{T}_i} \to k(\overline{s})$. The face $F'_i := f^{-1}(k(\overline{s})^*)$ is the preimage of F_i in $H_{\mathscr{T}_i} \to H^{\mathrm{sat}}_{\mathscr{T}_i}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F'_i \arrow[r] \arrow[d] & H_{\mathscr{T}_i} \arrow[r] \arrow[d] & H_{\mathscr{P}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_i \arrow[r] & H^{\mathrm{sat}}_{\mathscr{T}_i} \arrow[r] & H^{\mathrm{sat}}_{\mathscr{P}} \arrow[r] & 0
\end{tikzcd}$$

By Corollary 2.52, the bottom sequence is exact. We claim that the top sequence is also exact. Since $H_{\mathscr{P}}$ is integral, $H_{\mathscr{P}} \to H^{\mathrm{sat}}_{\mathscr{P}}$ is injective. It follows that F'_i is mapped to 0 in $H_{\mathscr{P}}$. By the definition of F'_i, the left square is cartesian. Therefore, the top sequence is exact except possibly at $H_{\mathscr{P}}$. By ([Ols08] Corollary 3.1.22), the quotient $H_{\mathscr{T}_i}/F'_i$ is actually $H_{\mathscr{P}}$. It follows that $H_{\mathscr{P}} \to H^{\mathrm{sat}}_{\mathscr{P}} \oplus F^{\mathrm{gp}}_i \to k[S_\tau]$ is a fine chart. This log structure is independent of the choice of splittings and the choice of i.

For the same reason, we can use the chart $S(Q) \rtimes H_{\mathscr{P}} \to k[S(Q) \rtimes_i S_\tau]$ for the log structure from $(\mathcal{X}_i, P_i)$. We can embed them into charts $(S(Q) \rtimes H_{\mathscr{P}}) \oplus S^*_\tau \to k[S(Q) \rtimes_i S_\tau]$. The isomorphism $\varphi_{\mathscr{T}_1\mathscr{T}_2}$ induces an isomorphism between the pre-log structures $(S(Q) \rtimes H_{\mathscr{P}}) \oplus S^*_\tau \to k[S(Q) \rtimes_i S_\tau]$.

For any geometric point $\overline{s}$ in the interior of the closed orbit corresponding to τ, $H_{\mathscr{P}}$ is sharp, and $H_{\mathscr{P}} \backslash \{0\}$ is mapped to 0 in $k(\overline{s})$. $\square$

It follows that $\varphi_{\mathscr{T}_1\mathscr{T}_2}$ is a morphism in $\mathscr{K}_Q(U_\tau)$. Use the composition rule in the stack $\mathscr{K}_Q$,

Proposition 2.54. *We have an injective morphism $R_{12} \to U_{C(\mathscr{T}_1)} \times_{\mathscr{K}_Q} U_{C(\mathscr{T}_2)}$ as a morphism between sheaves over the big étale site.*

Let's also denote $R_{\mathscr{T}}$ by $R_{\mathscr{T}\mathscr{T}}$ and define $R := \coprod_{\mathscr{T}_1,\mathscr{T}_2} R_{\mathscr{T}_1\mathscr{T}_2}$. We have two étale surjective morphisms $s, t : R \to U$ and the morphism $(s,t) : R \to U \times U$ is finite. Let $R \times_{s,U,t} R$ be X_2. Define a morphism $\mu : X_2 \to R$ as follows. For convenience, we also use the following notations. R_{11} or R_{22} means $R_{\mathscr{T}}$, R_{12} or R_{23} means $R_{\mathscr{T}_1\mathscr{T}_2}$ with $\mathscr{T}_1 \neq \mathscr{T}_2$. X_2 is a disjoint union of the following different types.

$$R_{11} \times_U R_{11}, R_{21} \times_U R_{11}, R_{11} \times_U R_{12}, R_{12} \times_U R_{23}.$$

For $R_{11} \times_U R_{11}$, denote the corresponding étale chart by U_1 and the finite group scheme by G_1. $R_{11} = U_1 \times G_1$. Define $\mu : R_{11} \times_U R_{11} \to R_{11}$ by,

$$R_{11} \times_U R_{11} = (U_1 \times G_1)_s \times_{U_1,t} U_1 \times G_1 \cong G_1 \times (G_1 \times U_1) \xrightarrow{m} G_1 \times U_1 \cong R_{11}.$$

The isomorphism "$\cong$" is due to the fact that s is the projection. m is the multiplication for the group scheme G_1.

Let τ be the intersection of $\mathscr{T}_1$ and $\mathscr{T}_2$. Similarly for $R_{21} \times_U R_{11}$ and $R_{11} \times_U R_{12}$, the morphism μ is defined by

$$R_{11} \times_U R_{12} = (U_1 \times G_1)_s \times_{U_1,t} (G_1 \times U_\tau \times G_2) \cong G_1 \times G_1 \times U_\tau \times G_2 \xrightarrow{m} G_1 \times U_\tau \times G_2 = R_{12},$$

and

$$
\begin{aligned}
R_{21} \times_U R_{11} &= (G_2 \times U_\tau \times G_1)_s \times_{U_1,t} (U_1 \times G_1) \\
&= G_2 \times (U_\tau \times G_1)_s \times_{U_1,t} (U_1 \times G_1) \\
&\cong G_2 \times (U_\tau \times_{U_1,t} (U_1 \times G_1))_s \times_{U_1,t} (U_1 \times G_1) \\
&\cong G_2 \times U_\tau \times_{U_1,t} ((U_1 \times G_1)_s \times_{U_1,t} (U_1 \times G_1)) \\
&\to G_2 \times U_\tau \times_{U_1,t} (U_1 \times G_1) \\
&\cong G_2 \times U_\tau \times G_1 = R_{21}.
\end{aligned}
$$

Here we used μ for R_{11} and the automorphism $(g,x) \mapsto (g,gx)$ of $G_1 \times U_1$ to replace s by t.

For $R_{12} \times_U R_{23}$, we can play the trick to compose the morphisms from G_i. So it surffices to define the morphism $U_\tau \times_{U_2} U_\upsilon \to U_\rho$, where τ is the intersection of $C(\mathscr{T}_1)$ and $C(\mathscr{T}_2)$, υ is the intersection of $C(\mathscr{T}_2)$ and $C(\mathscr{T}_3)$, and ρ is the intersection of $C(\mathscr{T}_1)$ and $C(\mathscr{T}_3)$. The point is the intersection $\varrho = \overline{\tau} \cap \overline{\upsilon}$ is equal to the intersection $C(\mathscr{T}_1) \cap C(\mathscr{T}_2) \cap C(\mathscr{T}_3)$. The affine toric variety $U_\tau \times_{U_2} U_\upsilon$ is defined by the cone $\varrho^\vee$ with the lattice $\mathbb{L}_\tau + \mathbb{L}_\upsilon = \mathbb{L}_1 + \mathbb{L}_2 + \mathbb{L}_3$. The cone $\rho^\vee$ is contained in the cone $\varrho^\vee$ and the lattice $\mathbb{L}_\tau + \mathbb{L}_\upsilon$ is finer than the lattice $\mathbb{L}_\rho$. Therefore there is a natural inclusion between the monoids which induces a morphism $U_\tau \times_{U_2} U_\upsilon \to U_\rho$. This gives the desired morphism μ.

In summary, we have defined a morphism $\mu : X_2 \to R$. Define $\epsilon : U \to R$ by the identity section $U_{C(\mathscr{T})} \to U_{C(\mathscr{T})} \times G = R_{\mathscr{T}}$. Let $\mathcal{C}$ be a category in which finite fiber products always exist. Recall the definition of a groupoid in $\mathcal{C}$ from ([FC90] Page 20).

Definition 2.55. A groupoid in $\mathcal{C}$ consists of the following data:

1. Two objects X_0, X_1 in $\mathcal{C}$ and two morphisms $s, t : X_1 \to X_0$.

2. a morphism $\mu : X_2 := X_s \times_{X_0,t} X_1 \to X_1$.

Let p_1, p_2 be the first and second projections from X_2 to X_1. These data should satisfy the following conditions:

a) The diagrams below are Cartesian.

$$\begin{tikzcd} X_2 \arrow[r, "\mu"] \arrow[d, "p_1"'] & X_1 \arrow[d, "t"] \\ X_1 \arrow[r, "t"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "\mu"] \arrow[d, "p_2"'] & X_1 \arrow[d, "s"] \\ X_1 \arrow[r, "s"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_1 \arrow[d, "t"] \\ X_1 \arrow[r, "s"] & X_0. \end{tikzcd}$$

b) The following diagram commutes.

$$\begin{tikzcd} X_1 \times_{s,t} X_1 \times_{s,t} X_1 \arrow[r, "\mu\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times\mu"'] & X_1 \times_{s,t} X_1 \arrow[d, "\mu"] \\ X_1 \times_{s,t} X_1 \arrow[r, "\mu"] & X_1. \end{tikzcd}$$

c) There exists a morphism $\epsilon : X_0 \to X_1$ such that $s \circ \epsilon = t \circ \epsilon = \mathrm{Id}$.

Proposition 2.56. *The data* $(U, R, s, t, \mu, \epsilon)$ *is a groupoid in* (Sch/k).

Proof. First, the third diagram in a) is automatically Cartesian by the definition of X_2. For the rest conditions, it is equivalent to show that for any

$S \in (\mathrm{Sch}/k)$, the data $(U(S), R(S), s, t, \mu, \epsilon)$ is a groupoid in sets. Therefore we check a), b), and c) for S-points. Divide into four different cases. For $R_{11} \times_U R_{11}, R_{21} \times_U R_{11}$ and $R_{11} \times_U R_{12}$, it reduces to the group actions on sets. For $R_{12} \times_U R_{23}$ case, recall that $U_\tau \times_{U_2} U_\upsilon$ is an affine toric variety defined by the monoid $\varrho^\vee \cap (\mathbb{L}_1 + \mathbb{L}_2 + \mathbb{L}_3)$. It follows that the following diagrams are Cartesian.

$$\begin{tikzcd} U_\tau \times_{U_2} U_\upsilon \arrow[r, "\mu"] \arrow[d, "p_1"'] & U_\rho \arrow[d, "t"] \\ U_\tau \arrow[r, "t"] & U_1 \end{tikzcd} \qquad \begin{tikzcd} U_\tau \times_{U_2} U_\upsilon \arrow[r, "\mu"] \arrow[d, "p_2"'] & U_\rho \arrow[d, "s"] \\ U_\upsilon \arrow[r, "s"] & U_3 \end{tikzcd}$$

Then we can use S-points and check the diagrams for sets. □

By ([LMB00] (3.4.3)), one can associate a stack $[U/R]$ with respect to the étale topology on (Sch/k). Denote $[U/R]$ by $\mathscr{T}_Q$.

Theorem 2.57. *The stack $\mathscr{T}_Q$ is a proper Deligne–Mumford stack with finite diagonal. It admits a coarse moduli space $\mathcal{T}_Q$.*

Proof. Since s and t are both étale, and $(s,t) : R \to U \times U$ is finite, by ([LMB00] Proposition (4.3.1)), $\mathscr{T}_Q$ is a Deligne–Mumford stack, and the canonical morphism $p : U \to \mathscr{T}_Q$ is an étale presentation. Since being finite is a property stable under base change and local in the étale topology on target (loc. cit. (3.10)), consider the cartesian diagram

$$\begin{tikzcd} U \times_{\mathscr{T}_Q} U \arrow[r, "i"] \arrow[d] & U \times U \arrow[d] \\ \mathscr{T}_Q \arrow[r, "\Delta"] & \mathscr{T}_Q \times \mathscr{T}_Q \end{tikzcd}$$

By (loc. cit. (3.4.3)), the fiber product $U \times_{\mathscr{T}_Q} U$ is isomorphic with R and the morphism i is the finite morphism $R \to U \times U$. This implies that the diagonal Δ is finite. By ([Ols08] Theorem 1.4.2), $\mathscr{T}_Q$ admits a coarse moduli space $\mathcal{T}_Q$. Since Δ is proper, $\mathscr{T}_Q$ is separated. Since U is quasi-compact, $\mathscr{T}_Q$ is quasi-compact. By the paragraph after 1.3 in [Ols05], we can use the valuative criterion of properness for discrete valuation rings . Furthermore, we can assume that the DVR (R, K) is complete with respect to the maximal ideal $\mathfrak{m}$, and the generic point η is in the open substack that corresponds to geometrically irreducible fiber X_η. After an étale base change, we can assume that X_η admits a K-point in the open torus orbit. The proof essentially follows from the proof of ([Ale02] Theorem 2.8.1) or ([Ols08] Lemma 3.7.8). Choose a uniformizer s. Let $\mathscr{P}$ be the decomposition, and $\psi : Q(\mathbf{Z}) \to \mathbf{Z}$ be the integral valued function obtained in the above proofs. Then, after a finite base change if necessary, $\psi \in C(\mathscr{P}) \cap \mathrm{Pic} \times \mathbf{Z}^{I_\emptyset}$. This defines a morphism $\operatorname{Spec} R \to U_{C(\mathscr{P})}$. The pull back of the standard family extends X_η to the whole $\operatorname{Spec} R$. $\square$

Lemma 2.58. *Denote the composition of morphism in the stack $\mathscr{K}_Q$ also by μ'. The following diagram commutes.*

$$\begin{tikzcd}
R_{12,s} \times_{U_2,t} R_{23} \arrow[r] \arrow[d, "\mu"'] & (U_{C(\mathscr{T}_1)} \times_{\mathscr{K}_Q} U_{C(\mathscr{T}_2)}) \times_U (U_{C(\mathscr{T}_2)} \times_{\mathscr{K}_Q} U_{C(\mathscr{T}_3)}) \arrow[d, "\mu'"] \\
R_{13} \arrow[r] & U_{C(\mathscr{T}_1)} \times_{\mathscr{K}_Q} U_{C(\mathscr{T}_3)}
\end{tikzcd}$$

Proof. The only nontrivial case to check is $1 \neq 2$ and $2 \neq 3$, and the morphism $R_{ij} \to U_{C(\mathscr{T}_i)} \times_{\mathscr{K}_Q} U_{C(\mathscr{T}_j)}$ is defined by the isomorphism $\varphi_{\mathscr{T}_i\mathscr{T}_j}$. However, the isomorphism $\varphi_{\mathscr{T}_i\mathscr{T}_j}$ is defined by the difference $g^{ij} = g_{\Psi,\mathscr{T}_i} - g_{\Psi,\mathscr{T}_j}$. Over

$U_{C(\mathscr{T}_1)} \times_U U_{C(\mathscr{T}_2)} \times_U U_{C(\mathscr{T}_3)}$, the cocycle condition

$$\varphi_{\mathscr{T}_i,\mathscr{T}_k} = \varphi_{\mathscr{T}_j\mathscr{T}_k} \circ \varphi_{\mathscr{T}_i\mathscr{T}_j},$$

is satisfied. This means exactly that the diagram above commutes in this case. □

The collection $\underline{\mathscr{U}} = \{U_{C(\mathscr{T}_i)}\}$ is an étale cover of $\mathscr{T}_Q$. For each $q \in Q(\mathbf{Q})$, the collection $\{g^{ij}(q) = g_{\Psi,\mathscr{T}_i}(q) - g_{\Psi,\mathscr{T}_j}(q)\}$ is a 1-cocycle in $\check{C}^1(\underline{\mathscr{U}}, \mathcal{O}^*)$. So it represents a line bundle $\mathcal{L}_q$. Lemma 2.58 means the gluing $\varphi_{\mathscr{T}_i\mathscr{T}_j}$ of the universal family is by twisting by the algebra of line bundles $\oplus\mathcal{L}_q$.

Proposition 2.59. *Let $R' := U \times_{\mathscr{K}_Q} U$. The morphism $R \to R'$ is injective as a morphism between sheaves over the big étale site.*

Proof. Combine Proposition 2.47 and Proposition 2.54. □

Proposition 2.60. *The 1-morphisms $F : U \to \mathscr{K}_Q$ and $R \to U \times_{\mathscr{K}_Q} U$ induces a proper 1-morphism $\overline{F} : \mathscr{T}_Q \to \mathscr{K}_Q$ between algebraic stacks.*

Proof. Let $R' := U \times_{\mathscr{K}_Q} U$. Consider the groupoid $(U, R', s', t', \mu', \epsilon')$ in (Sch/k) induced by the morphism $F : U \to \mathscr{K}_Q$. By Lemma 2.58, we have a morphism $\overline{F} : (U, R, s, t, \mu, \epsilon) \to (U, R', s', t', \mu', \epsilon')$ between groupoids in (Sch/k), and it induces a morphism $\overline{F} : [U/R] \to [U/R']$ by the universal property of stackification. By ([LMB00] Proposition (3.8)), $[U/R']$ is a substack of $\mathscr{K}_Q$. The composition gives $\overline{F} : \mathscr{T}_Q \to \mathscr{K}_Q$. Since $\mathscr{T}_Q$ and $\mathscr{K}_Q$ are both proper, $\overline{F}$ is proper by ([Ols13] Proposition 10.1.4 (iv)). □

Proposition 2.61. *The proper morphism $\overline{F}$ is surjective.*

Proof. Let $\mathscr{U}$ be the open dense locus of $\mathscr{K}_Q$ with trivial log structure. It is an open substack that classifying families all of whose geometric fibers are irreducible. The objects are polarized toric varieties X_Q with a torus embedding $T \to X_Q$ constructed from the integral polytope Q. The moduli is only for the divisor Θ. Fix any triangulation $\mathscr{T}$ and consider the open torus orbit of $U_{C(\mathscr{T})}$. This torus already parametrizes all pairs (X_Q, Θ) as above. Therefore, the image of $U_{C(\mathscr{T})}$ contains $\mathscr{U}$. Since $\overline{F}$ is proper and $\mathscr{U}$ is dense in $\mathscr{K}_Q$, $\overline{F}$ is surjective. In particular, $F : U \to \mathscr{K}_Q$ is surjective. By ([LMB00] Proposition (3.8)), $[U/R']$ is isomorphic to $\mathscr{K}_Q$. $\square$

Proposition 2.62. *The 1-morphism $\overline{F} : \mathscr{T}_Q \to \mathscr{K}_Q$ is representable. In particular $\overline{F}$ is proper as a representable morphism.*

Proof. By Proposition 2.59, $R \to R'$ is injective. Then $\overline{F}$ is representable by Lemma 2.63. By ([Ols13] Proposition 10.1.2), for a representable separated morphism of finite type, the two properness mean the same. $\square$

Lemma 2.63. *If $\overline{F}$ induces an injection $R \to R'$, then it is representable between algebraic stacks.*

Proof. We claim that the functor $\overline{F}$ is faithful. By the property of categories fibered in groupoids ([Ols13] Lemma 3.1.8), it suffices to prove that for any $S \in (\mathrm{Sch}/k)$, and $u, u' \in \mathscr{T}_Q(S)$, the map $\overline{F}_S : \mathrm{Isom}_{\mathscr{T}_Q(S)}(u, u') \to \mathrm{Isom}_{\mathscr{K}_Q(S)}(\overline{F}(u), \overline{F}(u'))$ is injective. Since $\mathrm{Isom}_{\mathscr{T}_Q(S)}(u, u')$ is a torsor over

$\mathrm{Aut}_{\mathscr{T}_Q(S)}(u)$, if it is not empty, it can be reduced to the case $u = u'$. These are global sections of presheaves $\underline{\mathrm{Isom}}(u, u)$ and $\underline{\mathrm{Isom}}(\overline{F}(u), \overline{F}(u))$. Since both $\mathscr{T}_Q$ and $\mathscr{K}_Q$ are algebraic stacks, both of the presheaves are sheaves for the étale topology on (Sch/S). So we can check the injectivity on some étale covering of S. Let $\xi : S' \to S$ be the pullback of $U \to \mathscr{T}_Q$ along $u : S \to \mathscr{T}_Q$.

$$\begin{tikzcd} S' \arrow[r, "v"] \arrow[d, "\xi"'] & U \arrow[d] \\ S \arrow[r, "u"] & \mathscr{T}_Q. \end{tikzcd}$$

By definition, there is an isomorphism $\rho : \xi^* u \to v$ in $\mathscr{T}_Q(S')$. Using ρ, replace $\xi^* u$ by v. Consider the Cartesian diagrams

$$\begin{tikzcd} \underline{\mathrm{Isom}}(v, v) \arrow[r] \arrow[d] & S' \arrow[d] \\ \mathscr{T}_Q \arrow[r, "\Delta"] & \mathscr{T}_Q \times \mathscr{T}_Q, \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "{(s,t)}"] \arrow[d] & U \times U \arrow[d] \\ \mathscr{T}_Q \arrow[r, "\Delta"] & \mathscr{T}_Q \times \mathscr{T}_Q. \end{tikzcd}$$

By the universal property of R, there is a 1-morphism $H : \underline{\mathrm{Isom}}(v, v) \to R$ such that the following diagram commutes,

$$\begin{tikzcd} \underline{\mathrm{Isom}}(v, v) \arrow[r] \arrow[d, "H"'] & S' \arrow[d, "{(v,v)}"] \\ R \arrow[r, "{(s,t)}"] & U \times U. \end{tikzcd}$$

This diagram is Cartesian. Similarly, after applying $\overline{F}$, we have the Cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}(\overline{F}(v), \overline{F}(v)) \arrow[r] \arrow[d, "\overline{F}(H)"'] & S' \arrow[d, "{(v,v)}"] \\ R' \arrow[r, "{(s,t)}"] & U \times U. \end{tikzcd}$$

Since the composition of two pullbacks is a pullback, the diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}(v,v) \arrow[r] \arrow[d, "H"'] & \underline{\mathrm{Isom}}(\overline{F}(v),\overline{F}(v)) \arrow[d, "\overline{F}(H)"'] \\ R \arrow[r] & R' \end{tikzcd}$$

is Cartesian. Then $R \to R'$ is injective implies that the map $\overline{F}_{S'}$: $\mathrm{Aut}_{\mathscr{T}_Q(S')}(v) \to \mathrm{Aut}_{\mathscr{K}_Q(S')}(\overline{F}(v))$ is injective. The claim that $\overline{F}$ is faithful is thus proved.

By ([dJea] Lemma 67.15.2), $\overline{F}$ is representable. $\square$

By construction, the algebraic stack $\mathscr{T}_Q$ is covered by open substacks $[U_{C(\mathscr{T})}/R_{\mathscr{T}}]$. The coarse moduli space $\mathcal{T}_Q$ is thus obtained by gluing the affine toric varieties $X_{C(\mathscr{T})}$ constructed by using the lattice $\mathbb{L}$. The gluing is by identifying different lattices in the same space $\mathbb{L}_{\mathbf{Q}}$. It is exactly the construction of the toric variety $X_{\Sigma(Q)}$ from the secondary fan $\Sigma(Q)$ and the lattice $\mathbb{L}^*$. Therefore $\mathcal{T}_Q \cong X_{\Sigma(Q)}$. Since $\mathscr{K}_Q$ is isomorphic to the main irreducible component of $\mathscr{T}\mathscr{P}^{fr}[Q]$ introduced in [Ale02] ([Ols08] Theorem 3.7.3), and the normalization of the irreducible main component of $\mathscr{T}\mathscr{P}^{fr}[Q]$ is isomorphic to $X_{\Sigma(Q)}$ ([Ale02] Corollary 2.12.3). The morphism induced by $\overline{F}$ between coarse moduli spaces is the normalization.

Remark 2.64. Strictly speaking, we need to check that the morphism between coarse moduli spaces is the compositions of the isomorphisms mentioned above. It can be done via the explicit description of the morphism in the proof of ([Ale02] Theorem 2.11.8).

Theorem 2.65 (The Main Theorem)**.** *Assume the lattice polytope Q contains a regular simplex. We have a compatification $\mathscr{T}_Q$ of the moduli space of toric pairs (X_Q, Θ). Over $\mathbf{Q}$, $\mathscr{T}_Q$ is a proper Deligne-Mumford stack of finite type, with finite diagonal. It admits a coarse moduli space $\mathcal{T}_Q \cong X_{\Sigma(Q)}$. Furthermore, there is a proper, surjective, representable 1-morphism $\overline{F}$ from $\mathscr{T}_Q$ to the moduli stack $\mathscr{K}_Q$ defined in [Ols08], that induces the normalization between the coarse moduli spaces.*

Remark 2.66. The construction of $\mathscr{T}_Q$ can be carried out over $\mathbf{Z}$. In that case, we only get an Artin stack with finite diagonal.

Since the $\mathbb{T}$-action on $\mathcal{L}$ is well-defined over $\mathscr{T}_Q$, the Fourier decomposition of the space of global sections $\Gamma(X, \mathcal{L}) = \oplus_{\omega \in I} \mathcal{V}_\omega$ is well-defined over the moduli space $\mathscr{T}_Q$. Then we have the Fourier decomposition of the section $\vartheta = \oplus_{\omega \in I} \vartheta_\omega$. If ω is in the interior of Q, the component ϑ_ω may vanish. This defines a stratification of the coarse moduli space $\mathcal{T}_Q = X_{\Sigma(Q)}$: For any subset $J \subset I$, the locus $\vartheta_\omega = 0$ for $\omega \in J$ is a locally closed subset of $\mathcal{T}_Q$. Each stratum is $T_{\mathbb{L}^*}$-invariant, because the action of $a \in T_{\mathbb{L}^*}$ is rescaling the ϑ_ω by $\Psi(\omega)(a) \in \mathbb{G}_m$. In particular, the locus $\mathcal{U}$ where none of the ϑ_ω is zero is a $T_{\mathbb{L}^*}$-invariant open subvariety of $\mathcal{T}_Q$.

Proposition 2.67. *The nondegenerate locus $\mathcal{U}$ is a toric variety projective over an affine toric variety. In particular, over $\mathbf{C}$, $\mathcal{U}$ is simply connected.*

Proof. Let Σ_M be the restriction of the secondary fan $\Sigma(Q)$ to the closure of the moving cone $\bigcup_{I \in \mathscr{T}} \overline{C}(\mathscr{T})$. We claim that $\mathcal{U}$ is the toric variety obtained by the fan Σ_M.

If $I \in \mathscr{P}$, since (X, Θ) is a stable toric pair, it is in $\mathcal{U}$. In particular each 0-stratum corresponding to the cone $C(\mathscr{T})$, where $\mathscr{T}$ is a triangulation containing I, is in $\mathcal{U}$. As a result, the orbits corresponding to faces of $\overline{C}(\mathscr{T})$ are contained in $\mathcal{U}$.

On the other hand, fix a point (X, Θ) in $\mathcal{U}$. Suppose the paving $\mathscr{P}$ associated to X doesn't contain all vertices in I. Consider a maximal cell $\sigma \in \mathscr{P}$ that contains a vertex $\omega \in I$ in its interior. Restrict ϑ to the component X_σ and denote it by ϑ_σ. $(X_\sigma, \vartheta_\sigma)$ is a point in the moduli of stable toric pairs $\mathcal{T}_\sigma$ associated to the lattice polytope σ. Since none of the components of ϑ_σ vanishes, $(X_\sigma, \vartheta_\sigma)$ is in the open torus orbit of $\mathcal{T}_\sigma$.Then we can construct a 1-parameter family, such that $(X_\sigma, \vartheta_\sigma)$ is a general fiber and the central fiber X_0, ϑ_0 has a paving $\mathscr{P}_0$ which contains all integral points of σ. By induction, we then construct a 1-parameter family, such that (X, Θ) is a general fiber and the central fiber (X', ϑ') has a paving $\mathscr{P}'$ which contains all integral points of Q. In other words, the $T_{\mathbb{L}^*}$-orbit of (X, Θ) corresponds to a face of the cone associated to (X', ϑ). Since the cone associated to (X', ϑ') is a face of $C(\mathscr{T})$, for some $I \subset \mathscr{T}$, the $T_{\mathbb{L}^*}$-orbit of (X, Θ) corresponds to a cone in Σ_M. The claim is proved.

Since the support of Σ_M is convex, $\mathcal{U}$ is projective over affine. $\square$

Remark 2.68. On the B-side, the secondary fan $\Sigma(Q)$ controls the degeneration of toric pairs. We also call the B-side the toric pair side. There is another interpretation in terms of algebraic cycles in $\mathbf{P}^{N-1}$. The integral polytope Q defines a finite morphism $X_Q \to \mathbf{P}^{N-1}$, whose image X'_Q is an algebraic cycle in $\mathbf{P}^{N-1}$. If the base is changed to $k = \mathbf{C}$, the action of the big torus $(\mathbf{C}^*)^N/\mathbf{C}^*$ on $\mathbf{P}^{N-1}$ induces an action on algebraic cycles. In [Ale02], Alexeev defines a map from $X_{\Sigma(Q)}$ to the parameter space of algebraic cycles induced by torus actions on X'_Q. Moreover, it is proved in [GKZ94], that the face poset of the secondary polytope $P(Q)$ is isomorphic to the poset of toric-specializations of $X'(Q)$. For a precise statement, see ([GKZ94] Chapter 8 Theorem 3.2).

Chapter 3

The Toroidal Compactification for the Coarse Moduli Space

For this chapter, the base ring k is $\mathbf{C}$.

3.1 The Toroidal Compactification: The Setup

3.1.1 The Coarse Moduli Space over C

Let $\mathrm{M}(g, \mathbf{C})$ denote the algebra of $g \times g$ complex matrices, I_g the identity element in $\mathrm{M}(g, \mathbf{C})$, and $\mathrm{M}(2g, \mathbf{R})$ the algebra of $2g \times 2g$ real matrices.

Definition 3.1. The complex domain,

$$\mathfrak{S}_g := \left\{ \tau \in \mathrm{M}(g, \mathbf{C});\ \tau = \tau^T, \Im(\tau) > 0 \right\},$$

is called the Siegel upper half space of degree g.

Definition 3.2.

$$\mathrm{Sp}(2g, \mathbf{R}) := \left\{ M \in \mathrm{M}(2g, \mathbf{R});\ M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M^T = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}$$

is called the real symplectic group.

More generally, for a non-degenerate skew-symmetric bilinear form E over $\mathbf{R}^{2g}$, we can define,

$$\mathrm{Sp}(E, \mathbf{R}) := \{M \in \mathrm{M}(2g, \mathbf{R});\ MEM^T = E\}.$$

Fix the notations

$$E = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \text{ where } \delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_g \end{pmatrix},$$

for positive integers δ_i, such that $\delta_i \mid \delta_{i+1}$, and $\prod_{i=1}^{g} \delta_i = d$.

The subgroup $\mathrm{Sp}(E, \mathbf{R})$ is conjugate to $\mathrm{Sp}(2g, \mathbf{R})$ in $\mathrm{GL}(2g, \mathbf{R})$ by the following element

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto M = \begin{pmatrix} \alpha & \beta \\ \gamma & \epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}^{-1} R \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{R}). \quad (3.1)$$

There are various arithmetic subgroups of $\mathrm{Sp}(2g, \mathbf{R})$ acting on $\mathfrak{S}_g$. We denote $\mathrm{Sp}(2g, \mathbf{Z})$ by $\Gamma(I)$. Likewise,

$$\Gamma(\delta) := \mathrm{Sp}(E, \mathbf{Z}), \quad (3.2)$$

$\Gamma(\delta)$ is an arithmetic subgroup of $\mathrm{Sp}(2g, \mathbf{R})$ through the map (3.1).

The following lemma is ([BL04] Lemma 17.2.3)[1]. We choose the convention that a vector in $H^1(A, \mathbf{C})$ be a column vector, and a period matrix is thus a $2g \times g$-matrix. The matrix multiplication on $\mathbf{R}^{2g}$ is from the right.

Lemma 3.3. *The following data are equivalent,*

a) a complex structure $J : \mathbf{R}^{2g} \to \mathbf{R}^{2g}, J^2 = -1$ such that $E = \Im(H)$, H a positive definite Hermitian form for this complex structure. The existence of H is equivalent to:

$$E(uJ, vJ) = E(u, v), \quad \forall u, v \in \mathbf{R}^{2g},$$
$$E(vJ, v) > 0, \quad \forall v \in \mathbf{R}^{2g} - \{0\}.$$

In this case, $H(u, v) = E(uJ, v) + iE(u, v)$. The set of such J is denoted by $\mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$.

b) a $g \times g$ complex symmetric matrix τ with $\Im(\tau)$ positive definite. The set of such τ is denoted by $\mathfrak{S}_g$.

The bijection is determined by the commutative diagram,

$$\begin{tikzcd} \mathbf{R}^{2g} \arrow[r, "\binom{\tau}{\delta}"] \arrow[d, "J"] & \mathbf{C}^{2g} \arrow[d, "iI_g"'] \\ \mathbf{R}^{2g} \arrow[r, "\binom{\tau}{\delta}"] & \mathbf{C}^{2g}. \end{tikzcd}$$

Moreover, the Lie group $\mathrm{Sp}(E, \mathbf{R})$ is acting on $\mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$ from the left by conjugation $J \mapsto RJR^{-1}$. The Lie group $\mathrm{Sp}(2g, \mathbf{R})$ is acting on $\mathfrak{S}_g$ by

[1]Notice that our convention is different from theirs by a transposition.

$\tau \mapsto (\alpha\tau + \beta)(\gamma\tau + \epsilon)^{-1}$. *The bijection* $\mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R})) \to \mathfrak{S}_g$ *is equivariant with respect to the action of Lie groups if we identify* $\mathrm{Sp}(E, \mathbf{R})$ *and* $\mathrm{Sp}(2g, \mathbf{R})$ *by the map* (3.1).

From now on, we identify $\mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$ with the Siegel upper half space $\mathfrak{S}_g$. The point is that the tropicalization is defined in terms of J, while the literature on toroidal compactifications uses $\mathfrak{S}_g$. We need to build a connection between these two in Sect. 3.1.3.

The group $\Gamma(\delta)$ is acting on the set of symplectic basis. Therefore, we have the action on $\mathfrak{S}_g$ from the left. For $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\delta)$, the action is

$$R(\tau) = (a\tau + b\delta)(c\tau + d\delta)^{-1}\delta. \tag{3.3}$$

For a polarized abelian variety A, we get $\Lambda = \mathrm{H}_1(A, \mathbf{Z})$, $V = (H^{1,0}(A))^*$, and polarization E. Over $\mathbf{C}$, the coarse moduli space of abelian varieties with polarization type δ is the quotient $\Gamma(\delta)\backslash\mathfrak{S}_g$ ([BL04] Theorem 8.2.6, and Remark 8.10.4).

3.1.2 The Toroidal Compactification: The Theory

The coarse moduli space $\mathcal{A}_{g,\delta,\mathbf{C}}$, as an arithmetic quotient of a bounded symmetric domain, admits a type of compactifications called the toroidal compactifications. We don't get into the details of the constructions, but fix the notations as we need. Our basic reference is [HKW93]. The theory can be divided into two parts. The first part is called the algebraic part. We construct a minimal compactification by adding boundary components, also called cusps. The stratification by the boundary components is then encoded in a simplicial object called the Tits building. The second part is the geometric part. It is the toric blow up of the minimal compactification[2] over the boundaries. The blow up is described by a collections of fans, which is some additional data. The collection of fans is called *admissible* if the blow-ups can be glued together. Therefore, to construct a toroidal compactification, we only need to construct an admissible collection of fans.

We outline the algebraic part first.

Definition 3.4.

$$\mathfrak{D}_g = \{Z \in \mathrm{M}(g, \mathbf{C}); Z = Z^T, I_g - Z\bar{Z} > 0\}.$$

The space $\mathfrak{S}_g$ is isomorphic to the bounded domain $\mathfrak{D}_g$ via the Cayley transformation

$$\tau \mapsto Z = (\tau - iI_g)(\tau + iI_g)^{-1}.$$

Take the closure $\overline{\mathfrak{D}}_g$ in $\mathrm{M}(g, \mathbf{C})$. The action of $\Gamma(\delta)$ can be extended to $\overline{\mathfrak{D}}_g$. $\overline{\mathfrak{D}}_g$ is stratified by locally closed subsets called the boundary components. To define the minimal compactification, we only need the boundary component defined by rational equations, called the rational boundary components. Let $\mathfrak{D}_g^{\mathrm{rc}}$ denote the union of all rational boundary components, the quotient $\Gamma(\delta)\backslash\mathfrak{D}_g^{\mathrm{rc}}$ is the minimal compactification. We also call the proper rational

[2]It is better known as the Satake compactification.

boundary components cusps. If the rational boundary component is of dimension n, it is called an n-cusp. We don't need the definitions in this paper, but the correspondence U, a map from the set of boundary components to the set of isotropic subspaces of $\mathbf{R}^{2g}$. If F is a boundary component, we use $U(F)$ to denote the correspondent isotropic real space. The basic result is

Proposition 3.5. *U is a bijection between the following objects:*

1. *a) rational boundary components F of dimension n,*

   b) rational E-isotropic subspaces $U(F) \subset \mathbf{R}^{2g}$ of dimension $g-n$.

2. *a) pairs of adjacent boundary components $F' \succ F$,*

   b) pairs of E-isotropic subspaces $U(F') \subsetneqq U(F)$.

We have $U(M(F)) = U(F) \cdot M^{-1}$, for $M \in \Gamma(\delta)$. Each rational boundary component is equivalent under $\mathrm{Sp}(E, \mathbf{Q})$ (left action) to one of the following boundary components,

$$F^{(g')} = \left\{ \begin{pmatrix} Z' & 0 \\ 0 & I_{g-g'} \end{pmatrix} ; Z' \in \mathfrak{D}_{g'} \right\}, \ 0 \leqslant g' \leqslant g.$$

$$U^{(g')} = U(F^{(g')}) = \left\{ (0\ x); 0 \in \mathbf{R}^{g+g'}, x \in \mathbf{R}^{g-g'} \right\}.$$

Remark 3.6. Notice that the map U depends on the choice of δ. If we denote the correspondent isotropic subspace in the principally polarized case by U_p, then $U\left(\begin{smallmatrix} I & 0 \\ 0 & \delta \end{smallmatrix}\right) = U_p$.

The geometric part of the compactification. The convention is to work with the arithmetic subgroups of $\mathrm{Sp}(2g, \mathbf{Q})$. We use the relation (3.1) to get the corresponding subgroups in $\mathrm{Sp}(E, \mathbf{R})$. For any boundary component F, the stabilizer $\mathcal{P}(F)$ is a parabolic subgroup. If $F' = M(F)$, for $M \in \mathrm{Sp}(2g, \mathbf{R})$, we have $U(F') = U(F) \cdot M^{-1}$ and $\mathcal{P}(F') = M\mathcal{P}(F)M^{-1}$. We decompose each $\mathcal{P}(F)$ into[3]

$$\mathcal{P} = \mathcal{G}_l \ltimes (\mathcal{G}_h \ltimes (\mathcal{V} \ltimes \mathcal{P}')).$$

The center of the unipotent radical $\mathcal{P}'$ is a real vector space. Its normalizer in $\mathrm{Sp}(2g, \mathbf{R})$ is $\mathcal{P}$, and $\mathcal{P}$ is acting on $\mathcal{P}'$ by the conjugation. The centralizer of $\mathcal{P}'$ is $\mathcal{G}_h \ltimes (\mathcal{V} \ltimes \mathcal{P}')$, so it is essentially the action of $\mathcal{G}_l$. The cone $\mathcal{C}(F)$ that supports the fan is a special orbit of this action in $\mathcal{P}'$, which is given by the Harish-Chandra map. While it is hard to describe the Harish-Chandra map in general, it is easy to write down everything for the special case $F^{(g')}$.

Example 3.7. *Over $F^{(g')}$, we have,*

1. $U^{(g')} = \left\{(0\ x); 0 \in \mathbf{R}^{g+g'},\ x \in \mathbf{R}^{g-g'}\right\}$,
2. $\mathcal{P}'^{(g')} = \left\{ [Q] := \begin{pmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & I_{g-g'} & 0 & Q \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & I_{g-g'} \end{pmatrix}; Q \in \mathrm{Sym}_{g-g'}(\mathbf{R}) \right\} \cong \mathrm{Sym}_{g-g'}(\mathbf{R})$,
3. $\mathcal{C}^{(g')} = \left\{[Q]; 0 < Q \in \mathrm{Sym}_{g-g'}(\mathbf{R})\right\}$,

[3]We drop the decoration "F" to simplify the notations.

4. $\mathcal{G}_l^{(g')} = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & (u^T)^{-1} & 0 & 0 \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & u \end{pmatrix} ; u \in \mathrm{GL}(g-g', \mathbf{R}) \right\} \cong \mathrm{GL}(g-g', \mathbf{R}),$

5. $\mathcal{G}_h^{(g')} = \left\{ \begin{pmatrix} A' & 0 & B' & 0 \\ 0 & I_{g-g'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{g-g'} \end{pmatrix} ; \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}(2g', \mathbf{R}) \right\},$

6. $\mathcal{V}^{(g')} = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & n^T \\ m & I_{g-g'} & n & 0 \\ 0 & 0 & I_{g'} & -m^T \\ 0 & 0 & 0 & I_{g-g'} \end{pmatrix} ; m, n \in \mathrm{M}((g-g') \times g', \mathbf{R}), mn^T = (mn^T)^T \right\},$

7. $\delta_{g'} = \begin{pmatrix} \delta_{g'+1} & & \\ & \ddots & \\ & & \delta_g \end{pmatrix},$

8. $\overline{P}^{(g')} \cong \mathrm{GL}(\delta_{g'}) := \big\{ u \in \mathrm{GL}(g-g', \mathbf{Z}); \delta_{g'} u \delta_{g'}^{-1} \in \mathrm{GL}(g-g', \mathbf{Z}) \big\}.$

The action of $\overline{P}^{(g')}$ *on the vector space* $\mathcal{P}'^{(g')}$ *is*

$$Q \mapsto (u^T)^{-1} Q u^{-1}.$$

Definition 3.8. For an arbitrary boundary component F, let $M \in \mathrm{Sp}(2g, \mathbf{R})$ such that $F = M(F^{(g')})$, define

$$\mathcal{C}(F) := M \mathcal{C}^{(g')} M^{-1}.$$

Remark 3.9. The definition of $\mathcal{C}(F)$ is independent of the choice of M.

Definition 3.10. Define

$$\mathfrak{D}(F) := \mathcal{U}(F)_{\mathbf{C}} \cdot \mathfrak{D}_g.$$

$$\mathfrak{D}_g = \mathrm{Sp}(2g, \mathbf{R}) / \mathrm{U}(g) \subset \mathfrak{D}(F) \subset \mathfrak{D}_g^c.$$

We use the notation $E = B \ltimes F$ to mean that E is a fiber bundle over B with the fiber F. The key observation is that $\mathfrak{D}(F) = (F \ltimes \mathcal{V}) \ltimes \mathcal{P}'_{\mathbf{C}}$ is a $\mathcal{P}'_{\mathbf{C}}$-principal bundle and $\mathfrak{D}_g = (F \ltimes \mathcal{V}) \ltimes (\mathcal{P}' + i\mathcal{C}(F))$ as a subset, i.e., from the perspective of F, $\mathfrak{D}_g$ is a family of tube domains of $\mathcal{C}(F)$ over $F \ltimes \mathcal{V}$. There exists a $\mathcal{P}(F)$-equivariant map Φ, such that the action of $\mathcal{P}(F)$ on $\mathcal{P}'(F)$ is the adjoint action Ad, and $\mathcal{C}(F) \subset \mathcal{P}'(F)$ is an orbit of $\mathcal{P}(F)$.

$$\begin{tikzcd}
\Phi : \mathfrak{D}(F) \arrow[r] & \mathcal{P}'(F) \\
\mathfrak{D}_g \arrow[r] \arrow[u, hook] & \mathcal{C}(F) \arrow[u, hook]
\end{tikzcd}$$

Fix an arithmetic subgroup Γ. For each rational boundary component F, define $P(F) = \mathcal{P}(F) \cap \Gamma$, and the decomposition

$$P(F) = G_l \ltimes (G_h \ltimes (V \ltimes P')),$$

with P' acting on the fiber $\mathcal{P}'_{\mathbf{C}}$ by translation, V acting on $\mathcal{V}$, and G_h acting on F. The principle is the normal groups act on the fibers, and the quotient groups act on the bases.

The additional data for the toric blowup is called an admissible collection of fans.

Definition 3.11 (the admissible fan)**.** Let F be a rational boundary component of $\mathfrak{D}_g$. A fan Σ in $\mathcal{P}'(F)$ is called admissible (for the construction of a partial compactification) if it satisfies the following three conditions:

a) $\bigcup_{\sigma\in\Sigma} = \mathcal{C}(F)^{\mathrm{rc}}$.

b) $M(\sigma) \in \Sigma$ for every $\sigma \in \Sigma$ and every $M \in \overline{P}(F)$.

c) There are only finitely many orbits of simplices in Σ under the action of $\overline{P}(F)$ on Σ, i.e., $\overline{P}(F)\backslash\Sigma$ is a finite set.

Definition 3.12. Let $\widetilde{\Sigma} = \{\Sigma(F)\}$ be a collection of fans $\Sigma(F) \subset \mathcal{P}'(F)$, where $\Sigma(F)$ is defined for every rational boundary component F of $\mathfrak{D}_g$. Then $\widetilde{\Sigma}$ is called an admissible collection (of fans) if it satisfies the following three condition:

a) $\Sigma(F) \subset \mathcal{P}'(F)$ is an admissible fan in the sense of the definition above for every rational boundary component F.

b) If $F' = M(F)$ for some $M \in \Gamma$, then $\Sigma(F') = M(\Sigma(F))$ as fans in the space $\mathcal{P}'(F') = M(\mathcal{P}'(F))$.

c) If $F' \succ F$ is a pair of adjacent rational boundary components, then the equality $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$ holds as fans in $\mathcal{P}'(F') \subset \mathcal{P}'(F)$.

By c) in the above definition, we can glue the fans $\Sigma(F)$ when the collection $\{\Sigma(F)\}$ is admissible. Denote the piecewise linear topological space by Ω_g^{tr}. Furthremore, by b), the group Γ is acting on Ω_g^{tr} and preserves the fan structure. Denote the quotient $\Gamma\backslash\Omega_g^{\mathrm{tr}}$ by $\overline{A}_{g,\Gamma}^{\mathrm{tr}}$. If $\Gamma = \Gamma(\delta)$, we use denote $\overline{A}_{g,\Gamma}^{\mathrm{tr}}$ by $\overline{A}_{g,\delta}^{\mathrm{tr}}$[4].

Let $F = F_\xi$. Take the quotient by $P'(F_\xi)$ first. Denote the torus $P'(F_\xi)\backslash\mathcal{P}'(F_\xi)_{\mathbf{C}}$ by $T_\xi := T_{\mathbb{L}^*}$, and $\Xi_\xi^a := P'(F_\xi)\backslash\mathfrak{D}(F_\xi), \Xi_{C,\xi}^a := P'(F_\xi)\backslash\mathfrak{D}_g$. Ξ_ξ^a is a principal T_ξ-bundle. Consider the induced action of $P'(F_\xi)$ on $\mathfrak{D}(F_\xi)$. We can identify the lattice $\mathbb{L}_\xi^*$ of cocharacters of the fiber T_ξ with $P'(F_\xi)$ canonically. Denote the image of the stabilizer of each fiber in $\mathrm{Aut}(P'(F_\xi))$ by $\overline{P}(F_\xi)$. By the property of Φ, $\overline{P}(F_\xi)$ is the image of $P(F_\xi)$ under the adjoint map Ad and $\overline{P}(F_\xi) \cong G_l$, which is independent of the fiber.

Use the fan $\Sigma(F_\xi)$, construct the torus embedding $(X_{\Sigma,\xi}, T_\xi)$, and take the associated bundle $\overline{\Xi}_\xi^{\mathrm{an}} := \Xi_\xi^{\mathrm{an}} \times^{T_\xi} X_{\Sigma,\xi}$. Take the interior of the closure of $\Xi_{C,\xi}^{\mathrm{an}}$, and denote it by $\Xi_{\Sigma,\xi}^{\mathrm{an}}$. This is the partial compactification over the cusp F_ξ. The collection $\widetilde{\Sigma}$ is admissible means that the partial compactifications are compatible with the group actions and the gluings.

Theorem 3.13 (Toroidal Compactification)**.** *If we have an admissible collection of fans, then we can construct a toroidal comapctification of the complex analytic space* $\Gamma(\delta)\backslash\mathfrak{D}_g$.

[4]the superscript "tr" is for tropicalization

For the more general statement of the toroidal compactification, see [HKW93].

3.1.3 A Tropical Interpretation of the Cones

In this section, we identify the cone $\mathcal{C}(F)$ with the moduli of polarized tropical abelian varieties. Fix a rational isotropic subspace U. We define the tropicalization in the direction of U first.

Lemma 3.14. $V = UJ \oplus U^\perp$.

Proof. If $u \in UJ \cap U^\perp$, then $uJ \in U$. Since $u \in U^\perp$, $E(uJ, u) = 0$. $E(\cdot J, \cdot)$ is nondegenerate, so $u = 0$, and $UJ \cap U^\perp = \{0\}$. Since $\dim UJ + \dim U^\perp = 2g$, $V = UJ \oplus U^\perp$. $\square$

Definition 3.15. For $J \in \mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$, define g_J to be the isomorphism $UJ \to V/U^\perp$ from Lemma 3.14. Define the map f_J,

$$f_J : U \to V/U^\perp,$$
$$\lambda \mapsto g_J(\lambda J).$$

By Lemma 3.14, f_J is an isomorphism. Denote the inverse by $\check{\phi} : V/U^\perp \to U$. Define $Y := \Lambda/\Lambda \cap U^\perp$. It is a full rank lattice in $V/U^\perp$. Consider the real torus $\check{B} := U/\check{\phi}(Y)$. Denote the lattice $U \cap \Lambda$ by X^*, and $\mathrm{Hom}(\Lambda \cap U, \mathbf{Z})$ by X. $\check{B}$ is a tropical torus with the lattice X^* in the tangent plane of $\check{B}$. The non-degenerate pairing $E(\cdot, \cdot)$ induces an injection $\phi : Y \to X$. The data ϕ is equivalent to the positive symmetric pairing $\check{g} = E(\cdot J, \cdot)$ on

U. Therefore ϕ is a polarization for the tropical torus $\check{B}$, and $(\check{B}, \check{\phi}, \phi)$ is a polarized tropical abelian variety in the sense of [MZ08]. Denote the image of $\phi(Y)$ by $\check{Y}$. Assume $X/\check{Y} \cong \mathbf{Z}/d_1 \times \mathbf{Z}/d_2 \times \ldots \times \mathbf{Z}/d_r$ for $d_i \mid d_{i+1}$. Let $\mathfrak{d}$ be the diagonal matrix

$$\mathfrak{d} := \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{pmatrix}.$$

$\mathfrak{d}$ is called the type of ϕ.

Corollary 3.16. $(\check{B}, \check{\phi}, \phi)$ is a r-dimensional polarized tropical abelian variety of type $\mathfrak{d}$.

Definition 3.17. Let $\check{Y} \subset X$ be two lattices of the same rank r. A basis of X $\{\alpha_1, \ldots, \alpha_r\}$ is called a compatible basis if there exists $\{d_1, \ldots, d_r\}$ such that $d_i | d_{i+1}$ and $\{d_1\alpha_1, \ldots, d_r\alpha_r\}$ is a basis of Y. In other words, there exists an isomorphism $X/\check{Y} \cong \mathbf{Z}/d_1 \times \mathbf{Z}/d_2 \times \cdots \times \mathbf{Z}/d_r$, such that α_i is sent to the generator of $\mathbf{Z}/d_i$.

Proposition 3.18. *Up to the action of* $\Gamma(\delta)$*, there is a unique rational boundary component that corresponds to the polarized tropical abelian varieties of type* δ*, and this is the orbit of* $F^{(0)}$*. For any* 0*-cusp, we have* $d_1 d_2 \cdots d_g = \delta_1 \delta_2 \cdots \delta_g$.

Proof. For the first statement, it suffices to find a symplectic basis $\{\lambda_i, \mu_j\}$ such that $\{\mu_j\}$ is a basis of $\Lambda \cap U$. Choose a compatible basis $\{y_j\}_{j \in 1, \ldots, g}$ of

X such that $\{x_i = \delta_i y_i\}$ is a basis of $\check{Y}$. Let's denote the sublattice generated by $\{x_{i+1}, \ldots, x_g\}$ by Y_i. Let $\{\mu_j\}$ be the dual basis of $\{y_j\}$. Consider the map $E(\cdot,\cdot)|_U : \Lambda \to X$, and lift x_1 to an element $\lambda_1 \in \Lambda$. So we have $E(\lambda_1, \mu_1) = \delta_1$ and $E(\lambda_1, \mu_j) = 0$ for $j \neq 1$. Now recall how δ_i is defined ([GH94] pp. 304-305.). The set of values $E(\Lambda, \Lambda)$ is an ideal in $\mathbf{Z}$ generated by δ_1. So δ_1 divides $E(\lambda, \mu)$ for any $\lambda, \mu \in \Lambda$. And we have a splitting $\Lambda = \{\lambda_1, \mu_1\} \oplus \{\lambda_1, \mu_1\}^{\perp}$. We denote $\{\lambda_1, \mu_1\}^{\perp}$ by Λ_1, and $E : \Lambda_1/\Lambda_1 \cap U \cong Y_1$. By the definition of δ_i we can do induction, and find all $\{\lambda_1, \lambda_2 \cdots, \lambda_g\}$ such that $\{\lambda_i, \mu_j\}$ is a symplectic basis of Λ.

For the second statement, choose a compatible basis $\{y_1, \ldots, y_g\}$ of X. Then there exists $\{u'_1, \ldots, u'_g\} \in \Lambda$ that are lifts of $d_i y_i$. Let $\{v_1, \ldots, v_g\} \subset U$ be the dual basis of $\{y_1, \ldots, y_g\}$. Then $\{u'_1, \ldots, u'_g, v_1, \ldots, v_g\}$ is a basis of Λ. With respect to this new basis, E is the matrix

$$\begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix},$$

where $\mathfrak{d}$ is the matrix (3.1.3), and S is some skew symmetric integral matrix (nonzero if $d \neq \delta$). Because the transformation matrix between two bases $\{u'_1, \ldots, u'_g,$
$v_1, \ldots, v_g\}$ and $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ is in $\mathrm{GL}(2g, \mathbf{Z})$, and has determinant ± 1, we have

$$d_1 d_2 \cdots d_g = \delta_1 \delta_2 \cdots \delta_g$$

by computing the determinant. □

Corollary 3.19. For any maximal isotropic subspace $U \subset \mathbf{R}^{2g}$, if $U \cap \Lambda$ has an isotropic complement, i.e. there is an isotropic subspace $U' \subset \mathbf{R}^{2g}$ such that $(U \cap \Lambda) \oplus (U' \cap \Lambda) = \Lambda$, then $U = U^{(0)} M^{-1}$ for some $M \in \Gamma(\delta)$.

Proof. If $U \cap \Lambda$ has an isotropic complement $U' \cap \Lambda$ in Λ, then choose a basis from $U \cap \Lambda$ and $U' \cap \Lambda$, we get a symplectice basis. $\square$

Definition 3.20. A maximal rational boundary component is called splitting if it is congruent to $F^{(0)}$. Otherwise it is called non-splitting.

Lemma 3.21. *With X and $\check{Y} \subset X$ fixed, the set of tropical abelian varieties is identified with the set of positive definite quadratic forms on $X_{\mathbf{R}}$, and is denoted by $\mathcal{C}(X)$.*

Proof. Since $\check{B}$ and $\phi : Y \to X$ are fixed, a polarized tropical abelian variety is equivalent to the data $\check{\phi} : Y \to U$, which is equivalent to the positive symmetric bilinear form $\langle \cdot, \check{\phi}\phi^{-1}(\cdot) \rangle$ on $X_{\mathbf{R}}$. $\square$

For a rational boundary component U, the map from $\mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$ to $\mathcal{C}(X)$ defined above is called the tropicalization, and is denoted by $\mathrm{Tr}(U)$ or Tr.

Definition 3.22. As in ([Ale02] 5.5.1), we define

$$\mathrm{GL}(X, Y) := \{u \in \mathrm{GL}(X); u\check{Y} \subset \check{Y}\}.$$

Consider X as a lattice in U^*. We identify $(S^2(U^*))^* = \Gamma^2(U) \subset U \otimes U$ with the vector space of quadratic forms on U^*. $\mathcal{C}(X)$ is an open cone in

$\Gamma^2(U)$. $\mathrm{GL}(X_{\mathbf{R}})$ has a natural representation $\bar{\rho}$ on $\Gamma^2(U)$. $\bar{\rho}$ is injective. $\mathcal{C}(X)$ is invariant under the action of $\bar{\rho}(\mathrm{GL}(X_{\mathbf{R}}))$. Thus $\mathrm{GL}(X,Y) \subset \mathrm{GL}(X_{\mathbf{R}})$ is acting on $\mathcal{C}(X)$. We have

Proposition 3.23. *Let $(\check{B}, \check{\phi}, \phi)$ and $(\check{B}, \check{\phi}', \phi)$ be tropical abelian varieties corresponding to $Q, Q' \in \mathcal{C}(X)$ respectively. $(\check{B}, \check{\phi}, \phi) \cong (\check{B}, \check{\phi}', \phi)$ as polarized tropical abelian varieties if and only if there is an element $u \in \mathrm{GL}(X,Y)$ such that $Q' = \bar{\rho}(u)(Q)$.*

Definition 3.24. A tropical abelian variety $(\check{B}, \check{\phi}, \phi)$ is called integral if $\check{\phi}(Y) \subset X^*$. A quadratic form $Q \in \Gamma^2(U)$ is called integral if the associated symmetric bilinear form B satisfies $B(\check{Y}, X) \in \mathbf{Z}$. The set of integral elements is a lattice, denoted by $\mathbb{L}^*$, in $\Gamma^2(U)$. The intersection of this lattice and $\mathcal{C}(X)$ is the set of integral tropical abelian varieties.

Note that $\Gamma^2 X \subset \mathbb{L}^*$. It is convenient to write down everything in terms of a fixed basis. Define

$$\mathrm{GL}(\mathfrak{d}) := \{u \in \mathrm{GL}(r, \mathbf{Z}) : \mathfrak{d} u \mathfrak{d}^{-1} \in \mathrm{GL}(r, \mathbf{Z})\}.$$

With respect to a compatible basis of X, $\mathrm{GL}(X,Y) = \mathrm{GL}(\mathfrak{d})^T$. $\mathcal{C}(X) \cong \{Q \in \mathrm{Sym}_g(\mathbf{R}); Q > 0\}$. The action is $Q \mapsto (u^T)^{-1} Q u^{-1}$. The set of integral tropical abelian varieties corresponds to $\{Q \in \mathcal{C}(X); Q\mathfrak{d} \in \mathrm{M}(r, \mathbf{Z})\}$. With respect to the corresponding basis of $\check{Y}$, $\mathrm{GL}(X,Y) \cong \mathrm{GL}(\mathfrak{d})$.

Given two cusps F and F', and an element $R \in \mathrm{Sp}(E, \mathbf{Q})$ such that $F' = R(F)$, R gives a map $\mathcal{C}(X) \to \mathcal{C}(X')$, still denoted by R, as follows:

Since $U' = UR^{-1}$, $R^{-1}|_U : U \to U'$ is a linear isomorphism. It induces an isomorphism $R^{-1}|_U \otimes R^{-1}|_U : \Gamma^2(U) \to \Gamma^2(U')$. This linear isomorphism sends $\mathcal{C}(X)$ to $\mathcal{C}(X')$.

Lemma 3.25. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_0(\mathrm{Sp}(E,\mathbf{R})) & \xrightarrow{R} & \mathcal{C}_0(\mathrm{Sp}(E,\mathbf{R})) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathcal{C}(X) & \xrightarrow{\quad R \quad} & \mathcal{C}(X') \end{array}.$$

The first line is the conjugation by R. Furthermore, if $R \in \Gamma(\delta)$, then $\mathrm{Tr}(J)$ *is isomorphic to* $\mathrm{Tr}(RJ)$ *as polarized tropical abelian varieties.*

Proof. First $R \in \mathrm{Sp}(E, \mathbf{Q})$. Notice that $E(\cdot J, \cdot)|_U = \langle \phi\check{\phi}^{-1}(\cdot), \cdot\rangle$ on U. For any $J \in \mathcal{C}_0(\mathrm{Sp}(E,\mathbf{R}))$, $R(J) = RJR^{-1}$. $\mathrm{Tr}(J)$ is represented by $E(\cdot J, \cdot)|_U$. The bilinear form $\mathrm{Tr}(R(J))$ is

$$E(\cdot R(J), \cdot)|_{R(U)} = E(\cdot RJR^{-1}, \cdot)|_{UR^{-1}} = E(\cdot RJ, \cdot R)|_{UR^{-1}}$$

This is the same as the bilinear form induced by R^{-1} in the definition.

Now $R \in \Gamma(\delta)$. Because $\cdot R^{-1}$ is a bijection $\mathbf{R}^{2g} \to \mathbf{R}^{2g}$, $(S \cap U)R^{-1} = SR^{-1} \cap UR^{-1}$ for any subset $S \subset \mathbf{R}^{2g}$. Since $R \in \Gamma(\delta)$, $(\Lambda \cap U)R^{-1} = \Lambda R^{-1} \cap UR^{-1} = \Lambda \cap U'$ and $(\Lambda \cap U^{\perp})R^{-1} = \Lambda \cap U'^{\perp}$. Consider the following commutative diagram for $J' = RJR^{-1}$,

$$\begin{array}{ccccc} \Lambda \cap U & \xrightarrow{\cdot J} & UJ \oplus U^{\perp} & \longrightarrow & V/U^{\perp} \\ \downarrow{\scriptstyle \cdot R^{-1}} & & \downarrow{\scriptstyle \cdot R^{-1}} & & \downarrow{\scriptstyle \cdot R^{-1}} \\ \Lambda \cap U' & \xrightarrow{\cdot J'} & UJ' \oplus U'^{\perp} & \longrightarrow & V/(U')^{\perp} \end{array}.$$

It shows that $f_{J'}(\Lambda \cap U') = (f_J(\Lambda \cap U))R^{-1}$ and $\check{B} \cong \check{B}'$ as tropical tori. The polarization is preserved because of the first part. $\square$

If $F = F'$, we can apply the above proof to $M \in \mathcal{P}(F)$, and get

Corollary 3.26. Fix a cusp F. The action of $\mathcal{P}(F)$ on $\mathcal{C}(X)$ is denoted by ρ_X. Then $\mathrm{Tr} : \mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R})) \to \mathcal{C}(X)$ is $\mathcal{P}(F)$-equivariant. The image $\rho_X(P(F))$ is inside $\mathrm{GL}(X, Y)$.

Let's consider the map $\mathrm{Tr} : \mathfrak{S}_g \to \mathcal{C}(X)$ for the cusps $F^{(g')}$. For $F^{(g')}$, $U^{(g')} = \{(0, x); 0 \in \mathbf{R}^{g+g'}, x \in \mathbf{R}^{g-g'}\}$. We have the natural basis for every lattice. Using the basis of X, that is $\{y_i\}_{i>g'}$, we regard $\mathcal{C}(X)$ as an open cone in $\mathrm{Sym}_r(\mathbf{R})$. Assume $\tau \in \mathfrak{S}_g$ corresponds to $J \in \mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R}))$. Since $E(\cdot J, \cdot) = \Re H$, and with respect to the coordinates $z_i = (x\tau + y\delta)_i$, H is $(\Im\tau)^{-1}$, the restriction of $E(\cdot J, \cdot)$ to $U^{(g')}$ is

$$E(\cdot J, \cdot)|_{U^{(g')}} = H|_{U^{(g')}} = \sum_{g'<i,j\leqslant g} \delta_i y_i ((\Im\tau)^{-1})_{ij} \delta_j y_j.$$

It follows that the matrix for $\mathrm{Tr}(\tau)$ is the inverse matrix of $T = ((\Im\tau)^{-1})_{g'<i,j\leqslant g}$ with respect to the basis of $\check{Y}$. We write τ in blocks, where τ_1 is a $g' \times g'$ matrix,

$$\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3^T & \tau_2 \end{pmatrix},$$

Lemma 3.27. *Assume given an invertible matrix and its inverse, both in blocks,*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

If A and D are both invertible, then

$$D^{-1} = d - ca^{-1}b.$$

Proof. We have

$$Aa + Bc = Id,$$

$$Ab + Bd = 0,$$

$$Ca + Dc = 0,$$

$$Cb + Dd = Id.$$

Therefore

$$D(d - ca^{-1}b) = Dd - Dca^{-1}b = Id - Cb + Caa^{-1}b = Id - Cb + Cb = Id.$$

□

Proposition 3.28. *With respect to the basis $\{\delta_i y_i\}_{i>g'}$ of $\check{Y}$,*

$$\mathrm{Tr}(\tau) = \Im\tau_2 - (\Im\tau_3^T)(\Im\tau_1)^{-1}\Im\tau_3.$$

Proof. Since τ is positive definite, τ_1 and τ_2 are positive definite. So we can apply the lemma. □

Now we can compare $\mathcal{C}(X)$ with $\mathcal{C}(F)$ for $F = F^{(g')}$.

Proposition 3.29. *For $F = F^{(g')}$, there exists an isomorphism $h(F) : \mathcal{P}'(F) \to \Gamma^2(U)$ restricting to a bijection $h(F) : \mathcal{C}(F) \to \mathcal{C}(X)$, such that the following* *diagram commutes,*

$$\begin{array}{ccc} \mathfrak{S}_g & \xrightarrow{=} & \mathcal{C}_0(\mathrm{Sp}(E,\mathbf{R})) \\ {\scriptstyle\Phi(F)}\downarrow & & \downarrow{\scriptstyle\mathrm{Tr}(F)} \\ \mathcal{C}(F) & \xrightarrow{h(F)} & \mathcal{C}(X) \end{array}$$

Moreover, the following statements are true.

a) Every map in the diagram is $\mathcal{P}(F)$-equivariant.

b) Tr *is surjective.*

c) Denote the induced isomorphism $\mathrm{Aut}(\Gamma^2(U)) \to \mathrm{Aut}(\mathcal{P}'(F))$ *by* ρ_h*, then* $\rho_h(\mathrm{GL}(X,Y)) = \overline{P}(F)$*.*

d) Thus $\rho_X(P(F)) = \mathrm{GL}(X,Y)$*.*

e) $P'(F) \cap \mathcal{C}(F)$ *is identified with the integral tropical abelian varieties by* $h(F)$*.*

Proof. We identify $\Gamma^2(U)$ with $\mathrm{Sym}_r(\mathbf{R})$ by the above basis $\{\delta_i y_i\}_{i>g'}$. We have also identified $\mathcal{P}'(F^{(g')})$ with $\mathrm{Sym}_r(\mathbf{R})$ by $[\cdot]$ in Example 3.7. $h(F)$ is defined to be the composition of these two isomorphisms. By ([Nam80], p. 31 ii), $\Phi(\tau) = \Im\tau_2 - (\Im\tau_3^T)(\Im\tau_1)^{-1}\Im\tau_3 = \mathrm{Tr}(\tau)$. Thus the diagram commutes.

Since Φ and Tr are both $\mathcal{P}(F)$-equivariant, and are both surjective, h is also $\mathcal{P}(F)$-equivariant, and we get a). For c), we have computed $\overline{P}(F) = \mathrm{GL}(\delta_{g'}) = \mathrm{GL}(X,Y)$ under the above identification. For e), $Q \in P'(F)$ if and only if $\delta_{g'}^{-1}Q \in \mathrm{M}(g-g',\mathbf{Z})$. Remember we are using the basis of $\check{Y}$. □

Assume $F' = M(F)$, for $M \in \mathrm{Sp}(2g, \mathbf{Q})$.

Lemma 3.30. *The following diagram*

$$\begin{tikzcd} \mathfrak{S}_g \arrow[r, "M"] \arrow[d, "\Phi"'] & \mathfrak{S}_g \arrow[d, "\Phi"'] \\ \mathcal{C}(F) \arrow[r, "M"] & \mathcal{C}(F'). \end{tikzcd}$$

commutes.

Proof. Recall the definition of Φ in [Nam80]. Embed $\mathfrak{S}_g \cong \mathcal{P}(F)/(\mathcal{P}(F) \cap K)$ into $\mathfrak{S}(F) \cong \mathcal{P}'(F)_{\mathbf{C}}\mathcal{P}(F)/(\mathcal{P}(F) \cap K)$. The map $\Phi : \mathfrak{S}(F) \to \mathcal{P}'(F)$ is the composition

$$\begin{array}{ccccc} \mathcal{P}'(F)_{\mathbf{C}}\mathcal{P}(F)/(\mathcal{P}(F) \cap K) & \longrightarrow & \mathcal{P}'(F)_{\mathbf{C}}\mathcal{P}(F)/\mathcal{P}(F) & \longrightarrow & \mathcal{P}'(F) \\ u \cdot p \mod \mathcal{P}(F) \cap K & \longmapsto & u \cdot p \mod \mathcal{P}(F) = u & \longmapsto & \Im u \end{array}$$

Assume $\tau = u \cdot p \mod \mathcal{P}(F) \cap K \in \mathfrak{S}_g$,

$$M(\tau) = M(u \cdot p \mod \mathcal{P}(F) \cap K) = MuM^{-1} \cdot MpM^{-1} \mod M(\mathcal{P}(F) \cap K)M^{-1}.$$

Because M is a real matrix,

$$\Phi(M\tau) = \Im MuM^{-1} = M \Im u M^{-1} = M\Phi(\tau).$$

$\square$

For any cusp F, there exists $M \in \mathrm{Sp}(2g, \mathbf{Q})$ such that $F = M(F^{(g')})$ for some g'. By definition, $\mathcal{P}'(F) = M\mathcal{P}'(F^{(g')})M^{-1}$. We can now define

$h(F) : \mathcal{P}'(F) \to \Gamma^2(U)$ to be the unique map that makes the following diagram commute,

$$\begin{tikzcd}
\mathcal{P}'(F^{(g')}) \arrow[r, "M"] \arrow[d, "h(F^{(g')})"'] & \mathcal{P}'(F) \arrow[d, "h(F)"'] \\
\Gamma^2(U^{(g')}) \arrow[r, "R"] & \Gamma^2(U).
\end{tikzcd}$$

The definition of $h(F)$ is independent of the choice of M, because $h(F^{(g')})$ is $\mathcal{P}(F^{(0)})$-equivariant. $h(F)$ is an isomorphism. Most of the statements for $h(F^{(g')})$ can be generalized to a general $h(F)$.

Proposition 3.31. *The following diagram commutes for any cusp F.*

$$\begin{tikzcd}
\mathfrak{S}_g \arrow[r, "="] \arrow[d, "\Phi(F)"'] & \mathcal{C}_0(\mathrm{Sp}(E, \mathbf{R})) \arrow[d, "\mathrm{Tr}(F)"'] \\
\mathcal{C}(F) \arrow[r, "h(F)"] & \mathcal{C}(X)
\end{tikzcd}$$

Moreover, the following statements are true.

a) Every map in the diagram is $\mathcal{P}(F)$-equivariant.

b) Tr *is surjective.*

c) $\rho_X(F) : \mathcal{P}(F) \to \mathrm{GL}(X_{\mathbf{R}})$ is surjective.

d) Denote the induced isomorphism $\mathrm{Aut}(\Gamma^2(U)) \to \mathrm{Aut}(\mathcal{P}'(F))$ by ρ_h, then $\overline{P}(F) \subset \rho_h(\mathrm{GL}(X, \check{Y}))$ is a subgroup of finite index.

Proof. We have the following diagram

$$\begin{tikzcd}
\mathcal{C}(F^{(g')}) \arrow[rr, "M"] \arrow[dd, "h(F^{(g')})"'] & & \mathcal{C}(F) \arrow[dd, "h(F)"] \\
& \mathfrak{S}_g \arrow[ul, "\Phi"] \arrow[ur, "\Phi"'] \arrow[dl, "\mathrm{Tr}"'] \arrow[dr, "\mathrm{Tr}"] & \\
\mathcal{C}(X^{(g')}) \arrow[rr, "R"] & & \mathcal{C}(X).
\end{tikzcd}$$

Imagine the diagram as a pyramid with five faces. The diagram concerning us is the triangular face on the right. It commutes because all the other faces commute and the arrows in the bottom square are all bijections. Then a), b) and c) follow.

For d). Since ρ_X is surjective and defined over $\mathbf{Q}$, by ([Bor69] 8.11), the image of the arithmetic subgroup $P(F)$ is an arithmetic subgroup of $\mathrm{GL}(X_{\mathbf{R}})$. So $\rho_X(P(F))$ is commensurable with $\mathrm{GL}(X, Y)$. Because $\rho_X(P(F))$ is a subgroup of $\mathrm{GL}(X, Y)$, it is a subgroup of finite index. $\overline{P}(F) = \mathrm{Ad}(\mathcal{P}(F)) = \rho_h \rho_X(P(F))$. $\square$

Let F be a 0-cusp that corresponds to a maximal isotropic space U. Fix a basis $\{v_1, \ldots, v_g\}$ of $U \cap \Lambda = X^*$. Take any $R \in \mathrm{Sp}(E, \mathbf{Q})$, such that $U = R \cdot U^{(0)} = U^{(0)} R^{-1}$, and R^{-1} maps $\{\mu_1, \ldots, \mu_g\}$ in $U^{(0)}$ to $\{v_1, \ldots, v_g\}$. Such R always exists. It suffices to find a rational Lagrangian complement L of U.

Let $Q \in \Gamma^2 U$ be represented as a matrix Q with respect to the dual basis of $\{v_1, v_2, \ldots, v_g\}$. Use the conjugacy of R^{-1} to map it into $\Gamma^2(U^{(0)})$. It is the quadratic form with the matrix Q under the basis $\{\mu_1, \mu_2, \ldots, \mu_g\}$. So it corresponds to

$$\begin{pmatrix} I_g & \delta Q \\ 0 & I_g \end{pmatrix} \in \mathcal{C}^{(0)}.$$

By definition of $h(F)$, we have

$$h(F)\left(R\begin{pmatrix} I_g & \delta Q \\ 0 & I_g \end{pmatrix}R^{-1}\right) = Q.$$

Since any other choice of R would be

$$R\begin{pmatrix} I_g & \delta Q' \\ 0 & I_g \end{pmatrix}$$

for some symmetric matrix Q', $h(F)$ is independent of the choice of R.

Extend the basis $\{v_1, \ldots, v_g\}$ of $U \cap \Lambda$ to a basis $\{u'_1, \ldots, u'_g, v_1, \ldots, v_g\}$ of Λ. Assume under this basis,

$$E = \begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix}.$$

Here S is an integral skew-symmetric matrix. Denote the transformation matrix of the bases by M'^{-1}. $M' \in \mathrm{GL}(2g, \mathbf{Z})$. Assume

$$\begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} M'^{-1} = R^{-1}$$

Since

$$M'^{-1}\begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}(M'^{-1})^T = \begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix},$$

we have

$$S = A^{-1}B\mathfrak{d} - (A^{-1}B\mathfrak{d})^T \tag{3.4}$$

$$A = \delta\mathfrak{d}^{-1} \tag{3.5}$$

We can always change R by

$$\begin{pmatrix} I_g & \delta Q' \\ 0 & I_g \end{pmatrix} R,$$

where Q' is a symmetric rational matrix. Therefore we can always choose R such that $A^{-1}B\mathfrak{d}$ is antisymmetric, that is

$$A^{-1}B\mathfrak{d} = \frac{1}{2}S \tag{3.6}$$

$$A = \delta\mathfrak{d}^{-1} \tag{3.7}$$

Corollary 3.32. The lattices $P'(F)$ and $\mathbb{L}^*$ agree. In particular, the set $\mathcal{C}(F) \cap P'(F)$ is the set of integral tropical abelian varieties for every F.

Proof. It suffices to prove it for maximal corank boundary components. Let U be a rational maximal isotropic subspace. Q corresponds to an element in $\mathcal{C}(F) \cap P'(F)$ if and only if

$$R \begin{pmatrix} I_g & \delta Q \\ 0 & I_g \end{pmatrix} R^{-1} \in \mathrm{GL}(2g, \mathbf{Z}).$$

Since $M' \in \mathrm{GL}(2g, \mathbf{Z})$, it is equivalent to

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix} \begin{pmatrix} I_g & \delta Q \\ 0 & I_g \end{pmatrix} \begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} \in \mathrm{GL}(2g, \mathbf{Z}).$$

This is true if and only if $\mathfrak{d}Q \in \mathbf{Z}$, which means Q represents an integral tropical abelian variety in $\mathcal{C}(X)$. $\square$

For any maximal corank rational boundary component F, we can also write down the homomorphsim $\rho_X : \mathcal{P}(F) \to \mathrm{GL}(X_{\mathbf{R}})$ explicitly. A typical element in $\mathcal{P}^{(0)}$ is like

$$\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta Q u \\ 0 & u \end{pmatrix},$$

for $u \in \mathrm{GL}(g, \mathbf{R})$ and Q symmetric. At $F^{(0)}$, the representation $\rho_{X^{(0)}}$ is

$$\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta Q u \\ 0 & u \end{pmatrix} \mapsto u.$$

Use the basis $\{v_1, v_2, \ldots, v_g\}$ and the transformation R as above, we have

$$\rho_X\left(R\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta Q u \\ 0 & u \end{pmatrix}R^{-1}\right) = u \tag{3.8}$$

If we use

$$R\begin{pmatrix} I_g & \delta Q' \\ 0 & I_g \end{pmatrix}$$

to define the map, it is

$$\rho_X\left(R\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta(Q + Q' - (u^{-1})^T Q' u^{-1})u \\ 0 & u \end{pmatrix}R^{-1}\right) = u \tag{3.9}$$

Therefore it is the same map.

Corollary 3.33. With respect to a compatible basis of X, the image $\rho_X(P(F))$ in $\mathrm{GL}(X,Y)$ are the matrices $u \in \mathrm{GL}(X,Y)$ with an additional condition: there exists a symmetric matrix Q such that

$$\frac{1}{2}(S\mathfrak{d}^{-1} - \mathfrak{d}u^T\mathfrak{d}^{-1}S\mathfrak{d}^{-1}u) + \mathfrak{d}Q \in \mathrm{M}(g, \mathbf{Z}).$$

Proof. u is in the image of $P(F)$ if and only if there exists a symmetric matrix Q such that

$$R\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta Q u \\ 0 & u \end{pmatrix} R^{-1} \in \mathrm{GL}(2g, \mathbf{Z}).$$

Equivalently,

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix}\begin{pmatrix} \delta(u^{-1})^T\delta^{-1} & \delta Q u \\ 0 & u \end{pmatrix}\begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} \in \mathrm{GL}(2g, \mathbf{Z}).$$

Choose R such that $A^{-1}B\mathfrak{d} = \frac{1}{2}S$ and compute, we have

$$u \in \mathrm{GL}(g, \mathbf{Z}), \tag{3.10}$$

$$\mathfrak{d}(u^T)^{-1}\mathfrak{d}^{-1} \in \mathrm{GL}(g, \mathbf{Z}), \tag{3.11}$$

$$\frac{1}{2}(\mathfrak{d}(u^{-1})^T\mathfrak{d}^{-1}S\mathfrak{d}^{-1} - S\mathfrak{d}^{-1}u) + \mathfrak{d}Qu \in \mathrm{M}(g, \mathbf{Z}). \tag{3.12}$$

□

3.2 Mirror Symmetry for Abelian Varieties

Warning: We use both X and Y to denote the complex tori and the lattices. We hope there is no confusion. : Fix a complex number $t \in \mathbf{H}$, and the polarization E on an abelian variety X, we get a symplectic manifold (X, Ω), with $\Omega = tE$ a complexified Kähler form. Any maximal corank rational isotropic subspace U is a linear Lagrangian for (X, Ω). Recall $Y := \Lambda/\Lambda \cap U$ is a lattice in V/U. Define the affine torus $B := (V/U)/Y$. A choice of U gives a Lagrangian fiberation $f : X \to B$. We use the construction in [Pol03] to define the mirror torus (Y_t, J_t). For $t = a + bi$, define $\Omega' = (ai - b)E$. Define $\phi : V \to V^*$ by $E(v, w) = \phi(v)(w)$. Define a complex structure J_Ω on $V \oplus V^*$ by requiring that the function

$$I_x(v, f) := \Omega'(x, v) + if(x)$$

is complex linear for each $x \in V$. It can be checked that

$$J_\Omega = \begin{pmatrix} b^{-1}a & -b^{-1}\phi^{-1} \\ (b + a^2 b^{-1})\phi & -ab^{-1} \end{pmatrix}$$

satisfies the condition. Denote the space of functions that vanish on U by $\operatorname{ann}(U)$. J_Ω preserves $U \oplus \operatorname{ann}(U)$, and thus descends to a complex structure J_t on

$$\check{V} := (V \oplus V^*)/(U \oplus \operatorname{ann}(U)) = V/U \oplus U^*.$$

Define the lattice $\Gamma := Y \oplus X \subset V/U \oplus U^*$. The complex torus $(\check{V}/\Gamma, J_t)$ is defined to be the mirror Y_t. There is a natural dual fiberation $\check{f} : Y_t \to B$. Let $\check{u}'_i \in Y$ be the image of $-u'_i$, and $\check{v}_i \in X$ be the dual of $v_i \in X^*$. Then

$$J_t \check{u}'_i = b^{-1} a \check{u}'_i - (b + a^2 b^{-1}) d_i \check{v}_i.$$

Define $\check{e}_i = d_i \check{v}_i$, then $\check{u}'_i = (a + bJ_t)\check{e}_i$. Use the basis $\{\check{e}_i\}$, $\{\check{u}'_i, \check{v}_j\}$, the period matrix is

$$\begin{pmatrix} tI_g \\ \mathfrak{d}^{-1} \end{pmatrix}.$$

Therefore, $Y_t = E_1 \times E_2 \times \ldots \times E_g$, where $E_i := \mathbf{C}/d_i^{-1}\mathbf{Z} + t\mathbf{Z}$ is an elliptic curve. We write E_i as $\mathbf{C}^*/(q^{\delta_i})^{\mathbf{Z}}$, for $q = e^{2\pi i t}$. Take Δ^* small enough, so that E_i has no complex multiplication for $q \in \Delta^*$. The fiber product of elliptic curves $\mathcal{Y}^* = \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \mathcal{E}_g \to \Delta^*$ is defined to be the mirror family for the maximal degeneration in the direction U.

While for any 0-cusp F, we can define such a mirror family, for general F, it is not the mirror family in the usual sense. If the fibers of $X \to B$ correspond to skyscraper sheaves, then there should exist a Lagrangian section which corresponds to the structure sheaf $\mathcal{O}_Y$. This is equivalent to ([Fuk02] Assumption 1.1), which says there is an isotropic subspace $L \subset V$ such that $(U \cap \Lambda) \oplus (L \cap \Lambda) = \Lambda$. By Corollary 3.19, then U is in the orbit of $F^{(0)}$.

3.2.1 Splitting Boundary Components

We consider the mirror symmetry for the boundary component $F^{(0)}$ first. Let L be the subspace generated by $\{\lambda_1, \ldots, \lambda_g\}$. Therefore L is a Lagrangian section for the fiberation $X \to B$, and we can identify V with $U \oplus V/U$. We use $\{\lambda_i, \mu_j\}$ as a basis for V.

Remark 3.34. By ([GLO01] proposition 9.6.1), in this case, for each $\tau \in \mathfrak{S}_g$, there exists ω_τ, such that (Y_t, ω_τ) is mirror symmetric to the algebraic pairs

(X_τ, tE) as defined in loc. cit. So our definition of the mirror agrees with the mirror in [GLO01].

A Chern class of a line bundle is given either by a hermitian form H on $\check{V}$, or equivalently, by a skew symmetric integral form of $\omega = \Im(H)$ on Γ. In terms of ω, the Riemann Conditions are

$$\omega(ix, iy) = \omega(x, y), \tag{3.13}$$

$$\omega(ix, x) > 0, \forall x \neq 0. \tag{3.14}$$

Assume, with respect to the basis $\{\check{\lambda}_1, \ldots, \check{\lambda}_g, \check{\mu}_1, \ldots, \check{\mu}_g\}$ of the lattice Γ,

$$\omega = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \in \mathrm{M}(2g, \mathbf{Z})$$

be the matrix of ω with respect to the basis $\{\check{\lambda}_i, \check{\mu}_j\}$. The Riemann conditions are

$$\begin{pmatrix} I_g & -\delta\bar{t} \end{pmatrix} \omega \begin{pmatrix} I_g \\ -\delta\bar{t} \end{pmatrix} = 0;$$

$$i \begin{pmatrix} I_g & -\delta\bar{t} \end{pmatrix} \omega \begin{pmatrix} -I_g \\ \delta t \end{pmatrix} > 0.$$

Since E_i are elliptic curves without complex multiplication, and $\Im(t) > 0$,

$$Q_1 = Q_4 = 0;\ Q_2\delta = -\delta Q_3;\ Q_2 = -Q_3^T;\ Q_3 > 0.$$

Let $Q = Q_3\delta^{-1}$, a positive definite symmetric matrix.

$$\omega = \begin{pmatrix} 0 & -\delta Q \\ Q\delta & 0 \end{pmatrix},$$

$$\omega = \sum_{i,j=1}^{g} -\delta_i Q_{ij}\, \mathrm{d}\check{x}_i \wedge \mathrm{d}\check{y}_j,$$

where $\{\check{x}_i, \check{y}_j\}$ are the dual of $\{\check{\lambda}_i, \check{\mu}_j\}$, and $\mathrm{d}\check{x}_i \wedge \mathrm{d}\check{y}_j = \mathrm{d}\check{x}_i \otimes \mathrm{d}\check{y}_j - \mathrm{d}\check{y}_j \otimes \mathrm{d}\check{x}_i$.

It follows that $X = \oplus \mathbf{Z}\check{\mu}_i$ is a maximally isotropic subgroup with respect to any ω. Therefore, we can do tropicalization $\Gamma/X \to U^*_{\mathbf{R}} = X_{\mathbf{R}}$. We find that the tropicalization is from the map $\phi : Y \to \check{Y}$.

$$\phi : \check{\lambda}_i \mapsto \delta_i \check{\mu}_i.$$

ω descends to a bilinear form $X \times Y \to \mathbf{Z}$. If we identify Y and $\check{Y}$ by ϕ, this bilinear form is symmetric and positive definite on $U^*_{\mathbf{R}}$, and thus corresponds to a positive quadratic form Q. It is equivalent to a data $\check{\phi} : Y \to X^*$. We see that the Kähler cone $\mathcal{K}(Y_t)$ is the set of all positive quadratic forms on X. Therefore, we have the mirror map $\mathcal{C}^{(0)} \cong \mathcal{C}(X) \cong \mathcal{K}(Y_t)$. Under this isomorphism, integral tropical abelian varieties are identified with the integral polarizations, i.e. $\mathrm{NS}(Y_t) = \mathbb{L}^*$. They are both represented by positive symmetric maps $\check{\phi}\phi^{-1} : X \to X^*$.

There is an explicit description of the map from a tropical abelian variety $\check{\phi}$ to a polarization on Y_t. It is the Fourier transform in [Pol03]. Fix $\check{\phi}$:

$Y \to X^* \subset U$ integral, extend it $\mathbf{R}$-linearly to a homomorphism $\check{\phi} : V/U \to U$. Denote the graph of $\check{\phi}$ by $\widetilde{L}_{\check{\phi}}$.

$$\widetilde{L}_{\check{\phi}} := \left\{(x, \check{\phi}(x)) \in V/U \times U\right\} = \{u + uJ \in V; u \in U\} \subset V.$$

Let $L_{\check{\phi}}$ be the image of $\widetilde{L}_{\check{\phi}}$ in X. It is a Lagrangian for (X, Ω). Since $L_{\check{\phi}}$ is constructed from the graph, and $\check{\phi}(Y) \subset X^*$, the fiberation $f : X \to B$ restricted to $L_{\check{\phi}} \to B$ is an isomorphism. Let $\iota : L_{\check{\phi}} \times_B Y \to X \times_B Y$ be the closed embedding. Since the fibers of $f : X \to B$ and $\check{f} : Y \to B$ are dual torus, and dual torus can be interpretated as the moduli space of unitary rank-1 local systems, there is a universal line bundle $\mathcal{P}$ with a universal unitary connection (not flat) on $X \times_B Y$. This is the real analogue of the Poincare line bundles. Therefore we can also construct a real analogue of the Fourier-Mukai transform. We can pull back this universal bundle $\mathcal{P}$ to $\iota^*\mathcal{P}$ over $L_{\check{\phi}} \times_B Y$.

Consider projections $p : L_{\check{\phi}} \times_B Y \to Y$ and $p_L : L_{\check{\phi}} \times_B Y \to L_{\check{\phi}}$. For any complex line bundle with a unitary flat connection $(\mathcal{L}, \nabla)$ over $L_{\check{\phi}}$, define the Fourier transform

$$\mathrm{Four}(L_{\check{\phi}}, \mathcal{L}) := p_*(p_L^* \mathcal{L} \otimes \iota^* \mathcal{P}).$$

Since p is an isomorphism, $\mathrm{Four}(L_{\check{\phi}}, \mathcal{L})$ has a unitary connection from those of $\mathcal{L}$ and $\mathcal{P}$. By ([Pol03] Theorem 6.3), with respect to the complex structure of Y_t, the $(0,1)$ part of the connection is integrable. Therefore $\mathrm{Four}(L_{\check{\phi}}, \mathcal{L})$ is a holomorphic line bundle for all Y_t.

Remark 3.35. For abelian varieties, the map from the Fukaya category to the derived category of coherent sheaves on the mirror, can be realized by this Fourier transform.

Since we only care about the first Chern class of $\mathrm{Four}(L_{\check{\phi}}, \mathcal{L})$, we assume $\mathcal{L}$ is trivial, and we have $p : L_{\check{\phi}} \times_B Y \cong Y$. Then we only need to study $\iota^*\mathcal{P}$. Consider the covers

$$\begin{tikzcd} \widetilde{L}_{\check{\phi}} \times_B Y \arrow[r, "g"] \arrow[d, "v"'] & U \times B \times (U^*/X) \arrow[d, "u"'] \\ L_{\check{\phi}} \times_B Y \arrow[r, "\iota"] & X \times_B \times Y. \end{tikzcd}$$

Use $\widetilde{L}_{\check{\phi}} \times_B Y \cong (V/U) \times_B B \times (U^*/X) \cong (V/U) \times (U^*/X)$, and the coordinates $(\check{x}, \check{y})$ on $(V/U) \times (U^*/X)$. The covering v is a quotient by the group Y, which acts on $\widetilde{L}_{\check{\phi}} \times_B Y$ by

$$\lambda \cdot (\check{x}, \check{y}) \mapsto (\check{x} + \lambda, \check{y}), \quad \forall \lambda \in Y.$$

The pullback $v^*\iota^*\mathcal{P} \cong g^*u^*\mathcal{P}$ is trivial on $\widetilde{L}_{\check{\phi}} \times_B Y$. Y is acting on a section s by

$$\lambda \cdot s(\check{x}, \check{y}) = \exp\left(-2\pi i \langle \check{\phi}(\lambda), \check{y} \rangle\right) s(\check{x} - \lambda, \check{y}).$$

With respect to a local trivialization, the connection is

$$\nabla = \mathrm{d} + 2\pi i \langle \mathrm{d}\check{y}, \check{\phi}(\check{x}) \rangle.$$

Here $\langle \mathrm{d}\check{y}, \check{\phi}(\check{x}) \rangle$ means as follows. In local coordinates $\check{y} = \sum_j \check{y}_j \check{\mu}_j$, $\check{x} = \sum_i \check{x}_i \check{\lambda}_i$, and $\check{\phi}(\check{x}) = \sum \check{x}_i (\delta Q)_{ij} \mu_j$, then

$$\langle \mathrm{d}\check{y}, \check{\phi}(\check{x}) \rangle := \sum_{ij} \check{x}_i (\delta Q)_{ij} \, \mathrm{d}\check{y}_j.$$

The curvature is

$$F_\nabla = \mathrm{d}(2\pi i \langle \mathrm{d}\check{y}, \check{\phi}(x)\rangle) = 2\pi i \sum_{ij} (\delta Q)_{ij}\, \mathrm{d}\check{x}_i \wedge \mathrm{d}\check{y}_j,$$

and the Chern class is

$$c_1(\mathrm{Four}(L_{\check{\phi}}, \mathcal{L})) = \frac{i}{2\pi} F_\nabla = \sum_{ij} -\delta_i Q_{ij}\, \mathrm{d}\check{x}_i \wedge \mathrm{d}\check{y}_j = \omega.$$

Therefore we have checked that

Corollary 3.36. The mirror map $\mathcal{C}(X) \to \mathcal{K}(Y_t)$, identifying Q, is given by the Fourier transform $\mathrm{Four}(L_{\check{\phi}}, \mathcal{L})$.

Remark 3.37. The Fourier transform is also defined in the non-splitting case.

Now we study the automorphism group of Y_t. Let $u \in \mathrm{GL}(g, \mathbf{C})$ be the representation of an automorphism under the basis $\{\check{e}_i\}$. Under the basis $\{\check{\lambda}_i, \check{\mu}_j\}$, the matrix is

$$\begin{pmatrix} u & 0 \\ 0 & \delta u \delta^{-1} \end{pmatrix}.$$

Therefore $u \in \mathrm{GL}(\delta)$ under the basis of Y. The group of automorphisms of Y_t is $\mathrm{GL}(X, Y)$. The action on the quadratic form Q is $Q' = (u^T)^{-1} Q u^{-1}$. It is identified with the action of $\mathrm{GL}(X, Y)$ on $\mathcal{C}(X)$.

We construct the relative minimal models $\mathcal{Y}/\Delta$ of the mirror family $\mathcal{Y}^*/\Delta^*$. A projective morphism $\check{\pi} : \mathcal{Y} \to \Delta$ is a relative minimal model if $\mathcal{Y}$ is $\mathbf{Q}$-factorial, terminal, and $K_{\mathcal{Y}}$ is $\check{\pi}$-nef. We use the construction in ([Mum72] Sect. 6). Replace Δ^* by a complete discrete valuation ring $(R, \mathfrak{m})$. Let K be the quotient field, and $\kappa = R/\mathfrak{m}$ the residue field. The base is $S = \operatorname{Spec} R$. The closed point is $s = \operatorname{Spec} \kappa$, and the generic point is $\eta = \operatorname{Spec} K$. Consider $X_{\mathbf{R}} = U^*$ as an affine plane of height 1 in the vector space $\mathbb{X}_{\mathbf{R}}$.

Definition 3.38. An integral paving $\mathscr{P}$ of U^* is called Y*-invariant* if $\mathscr{P}$ is invariant under the translation action of $\phi(Y)$ on U^*. Assume $\mathscr{P}$ is further a triangulation. It is called *minimal* if each vertex of X is a cell of $\mathscr{P}$, i.e. $X \subset \mathscr{P}$.

By ([CLS11] Exercise 8.2.14. & Proposition 11.4.12.), the minimal condition is necessary and sufficient for the relative complete model to be normal, $\mathbf{Q}$-factorial, $\mathbf{Q}$-Gorenstein, and terminal. Fix a Y-invariant integral paving $\mathscr{P}$. For each cell $\sigma \in \mathscr{P}$, construct the cone $C(\sigma)$ in $\mathbb{X}_{\mathbf{R}}$. The collection $\{C(\sigma)\}_{\sigma \in \mathscr{P}}$ forms a fan denoted by $\Sigma_{\mathscr{P}}$. Denote the infinite toric embedding $X_{\Sigma_{\mathscr{P}}}$ by $\widetilde{\mathfrak{Y}}_{\mathscr{P}}$. Since the fan $\Sigma_{\mathscr{P}}$ contains a basis of $\mathbb{X}$, $\widetilde{\mathfrak{Y}}_{\mathscr{P}}$ is simply connected as a complex analytic space. An embedding of $\check{\mathbb{T}} := \mathbb{G}_m^g/S$ into $\widetilde{\mathfrak{Y}}_{\mathscr{P}}$ is given by the vertical ray $C(\{0\})$ for $0 \in X$. To define a relatively ample line bundle over $\widetilde{\mathfrak{Y}}_{\mathscr{P}}$, with actions of Y and $\check{\mathbb{T}}$, we need additional data: a piecewise affine function φ on U^* such that

1) For each cell $\sigma \in \mathscr{P}$, $\varphi|_\sigma$ is affine. We say that φ is compatible with $\mathscr{P}$.

2) φ is strictly convex. That means the lower boundary of the graph of φ gives $\mathscr{P}$.

3) φ is Y-quasiperiodic.

4) φ is integral. That means for each $\sigma \in \mathscr{P}$, $\varphi|_\sigma$ is an element in $Aff(X, \mathbf{Z})$.

The set of functions which satisfy 1), 2) and 3) is denoted by $CPA^Y(\mathscr{P}, \mathbf{R})$. The set of functions which satisfy all 4 conditions is denoted by $CPA^Y(\mathscr{P}, \mathbf{Z})$.

By Lemma 1.3 in Appendix 1, φ should satisfy

$$\varphi(\alpha + \phi(y)) = \varphi(\alpha) + A(y) + \langle \alpha, \check{\phi}(y) \rangle.$$

$\check{\phi}$ is from a choice of a polarization ω, and we can always multiply it by some positive integer so that φ is integral. A is a quadratic function on Y such that

$$\langle \phi(y_1), \check{\phi}(y_2) \rangle = A(y_1 + y_2) - A(y_1) - A(y_2).$$

Assume

$$A(y) = \frac{1}{2}\langle \phi(y), \check{\phi}(y) \rangle + \frac{1}{2} l(\phi(y)).$$

for some $l \in U$.

The data φ defines an ample line bundle $\widetilde{\mathcal{L}}$ on $X_{\Sigma_{\mathscr{P}}}$. The action of Y

on $(X_{\Sigma_{\mathscr{P}}}, \widetilde{\mathcal{L}})$ is defined by

$$\begin{aligned}
\check{b} :& Y \times X^* \to K, \\
& (y, \mu) \mapsto q^{\langle \phi(y), \mu \rangle}; \\
\check{a} :& Y \to K^*, \\
& y \mapsto q^{A(y)}.
\end{aligned}$$

The action of $\check{\mathbb{T}}$ is defined by the grading.

Therefore the data $(\mathscr{P}, \varphi, \phi, \check{\phi}, A)$ gives rise to a relative complete model for the family $\mathcal{Y}^* \to \operatorname{Spec} K$. Thus we can take the quotient of $(\widetilde{\mathcal{Y}}_{\mathscr{P}}, \widetilde{\mathcal{L}})$ by Y as in ([Mum72] Sect. 3). The quotient $(\mathcal{Y}_{\mathscr{P}}, \mathcal{L})$ is a relative minimal model over S, which contains a semiabelian group scheme as a dense open subscheme. We call $\widetilde{\mathcal{Y}}_{\mathscr{P}}$ the universal covering of $\mathcal{Y}_{\mathscr{P}}$, because there exists a covering map in the complex topology.

We claim that an element in $\operatorname{Pic}(\mathcal{Y}_{\mathscr{P}})$ is represented by such an element φ, and the restriction map $r : \operatorname{Pic}(\mathcal{Y}_{\mathscr{P}}) \to \operatorname{NS}(Y_t)$ is given by $r(\varphi) \mapsto Q = \check{\phi} \circ \phi^{-1}$. The relations between the data is more explicit if we use the discrete Legendra transform, which, according Gross-Siebert program, is the mirror symmetry on the tropical level.

Define $B := U^*/\phi(Y) \cong (V/U)/Y$, and call it the dual complex. With $\check{\phi} : Y \to X^*$, B is a polarized tropical abelian. φ descends to an element in $\Gamma(B, \mathcal{PA} \,/\, \mathcal{A}\mathrm{ff})$. Here $\mathcal{PA}$ denotes the the sheaf of piecewise affine functions on an affine manifold, and $\mathcal{A}\mathrm{ff}$ denotes the sheaf of affine function. For an affine manifold, $\mathcal{PA}$ is the analogue of $\mathcal{K}^*$, and $\mathcal{A}$ff is the analogue of $\mathcal{O}^*$. So we call φ a tropical Cartier divisor on B.

Define $\check{B} := U/\check{\phi}(Y) \cong (V/U)/Y$. It has an integral affine structure induced by X^*, with a polarization $\phi : Y \to X$. We call $\check{B}$ the intersection complex for $\mathcal{Y}^*$. We see $\check{B}$ is actually a polarized abelian variety for the component $F^{(0)}$ and $(\check{B}, \check{\phi}, \phi) \in \mathcal{C}^{(0)}$.

The symmetric bilinear forms $\langle \phi(y_1), \check{\phi}(y_2) \rangle$ on Y extends to a metric g on B as well as a metric $\check{g}$ on $\check{B}$. Extend the quadratic function A from Y to U^*,

$$A_X(x) = \frac{1}{2}\langle x, \check{\phi} \circ \phi^{-1}(x) \rangle + \frac{1}{2} l(x).$$

By Lemma 1.3 in Appendix 1, $\varphi - A_X$ is $\check{Y}$-periodic.

We can identify the topological spaces B and $\check{B}$ by $-dA_X$. The metrics are identified, and is denoted by g. B and $\check{B}$ become two different tropical affine structures on the same torus $(V/U)/Y$.

We need to perform the discrete Legendre transform of $(B, \mathscr{P}, \varphi)$ to get $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. See [Gro12] for the local definition. Since there exists the universal cover U, we have a global construction. Take the tangent cone of φ at each vertex $(x, \varphi(x))$ for $x \in \mathscr{P}$. Since the subset above the graph of φ is a polyhedron Q_φ, the dual cones in $U \times \mathbf{R}$ form a fan in $U \times \mathbf{R}$. Take the intersection of the fan with $U \times \{1\}$, and get a decomposition $\check{\mathscr{P}}$. $\check{\mathscr{P}}$ is Y-invariant, and descends to a decomposition of $\check{B}$.

Definition 3.39. Define the piecewise affine function

$$\check{\varphi}(y) := -\inf_{x\in U^*}\{\varphi(x) + \langle x, y\rangle\}.$$

By definition, for any $y \in \check{\tau}$, we have

$$\inf_{x\in U^*}\{\varphi(x) + \langle x, y\rangle\} = \varphi(x') + \langle x', y\rangle \quad \forall x' \in \tau.$$

Locally, near $\check{\sigma}$, define $\check{\varphi}_{\check{\sigma}}(y) = \check{\varphi}(y+\check{\sigma}) - \check{\varphi}(\check{\sigma})$. If $y+\check{\sigma} \in \check{\tau}$ and $\tau \prec \sigma$, then

$$\check{\varphi}_{\check{\sigma}}(y) = -\inf_{x\in\sigma}\{\langle x, y\rangle\}.$$

Definition 3.40. The triplet $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is called *the discrete Legendre transform* of $(B, \mathscr{P}, \varphi)$.

Lemma 3.41.

$$\check{\varphi}(y + \check{\phi}(\lambda)) = \check{\varphi}(y) + \langle \phi(\lambda), y\rangle + A(\lambda),\ \forall \lambda \in Y.$$

Proof.

$$\begin{aligned}
\check{\varphi}(y + \check{\phi}(\lambda)) &= -\inf_{x\in U^*}\{\varphi(x) + \langle x, y + \check{\phi}(\lambda)\rangle\} \\
&= -\inf_{x\in U^*}\{\varphi(x) + \langle x, \check{\phi}(\lambda)\rangle + A(\lambda) + \langle x, y\rangle - A(\lambda)\} \\
&= -\inf_{x\in U^*}\{\varphi(x + \phi(\lambda)) + \langle x + \phi(\lambda), y\rangle\} + \langle \phi(\lambda), y\rangle + A(\lambda) \\
&= \check{\varphi}(y) + \langle \phi(\lambda), y\rangle + A(\lambda).
\end{aligned}$$

□

Corollary 3.42. The Y-action on the homogeneous coordinate ring is given as follows. Define $\zeta_\mu = \mathrm{X}^\mu q^{\check{\varphi}(\mu)}\theta$ for $\mu \in X^*$ to be a monomial section of $\widetilde{\mathcal{L}}$. Then

$$\begin{aligned} S_\lambda^*(\zeta_\mu) = \zeta_{\mu+\check{\phi}(\lambda)} &= \mathrm{X}^{\mu+\check{\phi}(\lambda)} q^{\check{\varphi}(\mu+\check{\phi}(\lambda))}\theta \\ &= \mathrm{X}^\mu q^{\check{\varphi}(\mu)}\theta \mathrm{X}^{\check{\phi}(\lambda)} \mathrm{X}^\mu(\lambda) q^{A(\lambda)} \\ &= \mathrm{X}^\mu(\lambda) q^{A(\lambda)} \mathrm{X}^{\check{\phi}(\lambda)} \zeta_\mu. \end{aligned}$$

Corollary 3.43. If $\psi = \varphi - \varphi'$ is an integral affine function, then the induced line bundles $\mathcal{L}$, and $\mathcal{L}'$ by φ and φ' are isomorphic on Y.

Proof. ψ defines an isomorphism between the homogeneous coordinate rings of $X_{\Sigma_{\mathscr{P}}}$. If $\psi(x) = \langle x, a\rangle + h$, the isomorphism is a translations by $-(a,h)$. Thus it is Y-equivariant by the action above. It descends to an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$. $\square$

It follows from Corollary 3.43 that we have a map $p : CPA^Y(\mathscr{P}, \mathbf{R})/Aff \to \mathcal{K}(\mathcal{Y}_{\mathscr{P}})$. If we use complex geometry, we can talk about the universal covering map $\Upsilon : \widetilde{\mathcal{Y}}_{\mathscr{P}} \to \mathcal{Y}_{\mathscr{P}}$. Since the Cartier divisors on $\widetilde{\mathcal{Y}}_{\mathscr{P}}$ are described by piecewise affine functions, the pull back of line bundles from $\mathcal{Y}_{\mathscr{P}}$ to $\widetilde{\mathcal{Y}}_{\mathscr{P}}$ defines $\Upsilon^* : \mathcal{K}(\mathcal{Y}_{\mathscr{P}}) \to CPA^Y(\mathscr{P}, \mathbf{R})/Aff$, such that $\Upsilon^* \circ p = \mathrm{Id}$. Therefore, the map $p : CPA^Y(\mathscr{P}, \mathbf{R})/Aff \to \mathcal{K}(\mathcal{Y}_{\mathscr{P}})$ is an injection.

Corollary 3.44. Regard $CPA^Y(\mathscr{P}, \mathbf{R})/Aff$ as a subset of $\mathcal{K}(\mathcal{Y}_{\mathscr{P}})$. The restriction map $r : CPA^Y(\mathscr{P}, \mathbf{R})/Aff \to \mathrm{NS}(Y_t)$ is given by,

$$\varphi \longmapsto \omega = Q = \check{\phi} \circ \phi^{-1}. \tag{3.15}$$

Proof. It follows from the computation in Corollary 3.42. $\square$

Assume that the set of the irreducible components of Y_s is indexed by $I = B(\mathbf{Z})$ and $|I| = \prod_i \delta_i = d$. Each irreducible component is a prime divisor $D_i, i \in I$.

Lemma 3.45. *Let $\mathscr{P}$ be minimal. We have the following exact sequence,*

$$0 \longrightarrow \mathbf{Z}^d/\mathbf{Z} \longrightarrow \mathrm{Pic}(\mathcal{Y}_{\mathscr{P}}) \xrightarrow{r} \mathrm{NS}(Y_t) \longrightarrow 0. \tag{3.16}$$

The mod $\mathbf{Z}$ is because $\sum_{i \in I} D_i = \breve{\pi}^{-1}(0) \equiv 0$.

In particular, $p : CPA^Y(\mathscr{P}, \mathbf{R})/Aff \to \mathcal{K}(\mathcal{Y}_{\mathscr{P}})$ is a bijection, and $\mathrm{Pic}(\mathcal{Y}_{\mathscr{P}})$ is identified with the space $\Gamma(B, \mathcal{PA}/\mathcal{A}ff)$.

Proof. For the exactness of the sequence (3.16), the only thing left to check is that $\sum_{i \in I} D_i = 0$ is the only relation in $\mathrm{Pic}(\mathcal{Y}_{\mathscr{P}})$ for $\{D_i\}$. If $\mathcal{Y}_{\mathscr{P}}$ is a surface, this follows from ([Sha13] Theorem 4.14). For the general case, assume we have a relation D_r. Intersect D_r with a generic hypersurface $\mathcal{S}'$ that is flat over Δ. Then, by induction, $D_r \cdot \mathcal{S}'$ is a multiple of $\sum_{i \in I} D_i \cdot \mathcal{S}'$. It follows that D_r is a multiple of $\sum_{i \in I} D_i$.

Since the sequence(3.16) is exact, $p \circ \Upsilon^* = \mathrm{Id}$, and $\mathrm{Pic}(\mathcal{Y}_{\mathscr{P}}) \cong \Gamma(B, \mathcal{PA}/\mathcal{A}ff)$. $\square$

The relative minimal model $\breve{\pi} : \mathcal{Y} \to \Delta$ is not unique, but any two models are isomorphic up to codimension 1, and are connected by a sequence of flops. See [Kaw08]. We need to study the Mori fan of the minimal model $\mathcal{Y}$. This fan is independent of the choice of the relative minimal model as in the GKZ case. The support is the rational closure of the big cone.

Lemma 3.46. *The pseudo-effective cone $\overline{\mathrm{Eff}}(\mathcal{Y}_{\mathscr{P}})$ is the closure of $r^{-1}(\mathcal{K}(Y_t))$.*

Proof. The restriction $r(D)$ of an effective divisor D is effective. Since Y_t is an abelian variety, $r(D)$ is in the closure of $\mathcal{K}(Y_t)$. Therefore, D is in the closure of $r^{-1}(\mathcal{K}(Y_t))$.

On the other hand, $\ker(r)$ are all effective, by adding $\sum_{i\in I} D_i$. Moreover, for any rational class $Q \in \mathcal{K}(Y_t)$, it is easy to construct an effective divisor φ_Q such that $r(\varphi_Q)$ is Q. For example, define $A_X = Q|_X$ and make it integral by multiplying a positive integer. A_X determines a Delaunay decomposition of (U^*, X). Define φ_Q to be the affine interpolation of A_X. $\square$

A real valued function ψ on X is called Y-quasiperiodic if it is a sum of a quadratic functions and a Y-periodic function ($\phi(Y) \subset X$). The set of Y-quasiperiodic functions over X is a vector space of finite dimension. Fix a Y-invariant integral triangulation $\mathscr{T}$ of $X_{\mathbf{R}}$. For any function ψ over X, we can define a piecewise affine function $g_{\psi,\mathscr{T}}$. For each vertex α of $\mathscr{T}$, define

$g_{\psi,\mathscr{T}}(\alpha) = \psi(\alpha)$. Then $g_{\psi,\mathscr{T}}$ is obtained by affine interpolation over each simplex of $\mathscr{T}$.

Definition 3.47. Let $\mathscr{T}$ be a Y-invariant integral triangulation. We shall denote by $\widetilde{C}^Y(\mathscr{T})$ the cone consisting of Y-quasiperiodic functions ψ over X with the following two properties:

a) The function $g_{\psi,\mathscr{T}}$ is convex.

b) For any $\alpha \in X$ but not a vertex of any simplex from $\mathscr{T}$, we have $g_{\psi,\mathscr{T}}(\alpha) \leqslant \psi(\alpha)$.

Define $C^Y(\mathscr{T})$ to be $\widetilde{C}^Y(\mathscr{T})/Aff$. If $\mathscr{P}$ is minimal, $C^Y(\mathscr{P}) = CPA^Y(\mathscr{P}, \mathbf{R})/Aff$.

Proposition 3.48. *Every Mori chamber is of the form $C^Y(\mathscr{T})$ for some triangulation $\mathscr{T}$. Every relative minimal model is isomorphic to $\mathcal{Y}_{\mathscr{P}'}$ for some minimal triangulation $\mathscr{P}'$.*

Proof. Fix $\mathscr{P}$ a minimal triangulation. For each Y-invariant triangulation $\mathscr{T}$, the rational map $\mathcal{Y}_{\mathscr{P}} \dashrightarrow \mathcal{Y}_{\mathscr{T}}$ is a contraction. It is a small contraction if and only if $\mathscr{T}$ is also minimal. We claim that each $C^Y(\mathscr{T})^\circ$ is contained in one Mori chamber.

For a $\mathbf{Q}$-Cartier $D \in C^Y(\mathscr{T})^\circ$ corresponding to a function ψ_D over X, decompose $\psi_D = \psi_A + \psi_E$, where $\psi_A = g_{\psi,\mathscr{T}}$ and $\psi_E = \psi_D - g_{\psi,\mathscr{T}}$. Since $g_{\psi,\mathscr{T}}$ is strictly convex, A is ample on $\mathcal{Y}_{\mathscr{T}}$. Then D defines the rational map $f_D : \mathcal{Y}_{\mathscr{P}} \dashrightarrow \mathcal{Y}_{\mathscr{T}}$ and $D = f_D^* A + E$ for E f_D-exceptional. It proves the claim.

By Lemma 3.45 and Lemma 3.46, we can identify $\overline{\mathrm{Eff}}(\mathcal{Y}_{\mathscr{P}})$ with the closed cone $\mathcal{C}^{+}$ in $\Gamma(B, \mathcal{PA}/\mathcal{A}ff)$, where the associated quadratic form Q is semi-positive definite. Denote the interior of $\mathcal{C}^{+}$ by $\mathcal{C}^{\circ}$, which corresponds to the big cone. The support of the Mori fan is the rational closure of $\mathcal{C}^{\circ}$, denoted by $\mathcal{C}^{\mathrm{rc}}$. However, by the same argument as in ([GKZ94] Chap.7, Proposition 1.5), the cones $C^{Y}(\mathscr{T})$ already form a fan supported on $\mathcal{C}^{\mathrm{rc}}$. Therefore, each cone in the Mori fan is of the form $C^{Y}(\mathscr{T})$, and the contraction is small if and only if $\mathscr{T}$ is minimal. □

For each $\mathcal{Y}_{\mathscr{P}}$, the complex $(B, \mathscr{P})$ is the dual complex $\mathcal{D}(Y_s)$ as defined in [dFKX14]. We have shown that the dual complexes $\mathcal{D}(Y_s)$ of all relative minimal models $\mathcal{Y}/\Delta$ are all homeomorphic. This is a special case of ([dFKX14] Proposition 11). Moreover, there is a canonical integral affine structure on $\mathcal{D}(Y_s)$ that is independent of the minimal model. Given a relative minimal model $\mathcal{Y}$, the integral affine manifold B can be recovered as follows. To define an affine structure, we only need to define affine functions over a union of two adjacent maximal cells. In dim $= 2$, suppose the two cells are $v_0v_1v_2$ and $v_0v_1v_3$, a function φ is affine if

$$((\varphi(v_1) - \varphi(v_0))D_{10} + (\varphi(v_2) - \varphi(v_0))D_{20} + (\varphi(v_3) - \varphi(v_0))D_{30}) \cdot D_{10} = 0$$

(It is well defined because of s.n.c., $D_{01}^2 + D_{10}^2 = -2$). Here we use the notation D_i means the irreducible components corresponding to v_i, D_{ij} means $D_i \cap D_j$ as a cycle in D_j, and D_{ijk} means $D_i \cap D_j \cap D_k$ as a cycle in D_k, and so on. In higher dimensions, for any two adjacent maximal cells

$v_0v_1v_2\ldots v_{g-1}v_g$ and $v_0v_1\ldots v_{g-1}v_{g+1}$, note that the span of $\{v_0, v_1, \ldots, v_{g-1}\}$ is a $(g-1)$-simplex, and $D_{12\ldots g-10} \cong \mathbf{P}^1$. The condition is

$$\left(\sum_{i=1}^{g+1}(\varphi(v_i) - \varphi(v_0))D_{i0}\right) \cdot D_{12\ldots g-10} = 0.$$

Since the central fiber is s.n.c., this condition is independent of the choice of v_0. To define integral structures, we have to specify the integral affine functions. An affine function φ is called integral, if $\varphi(v)$ is an integer for any vertex $v \in \mathcal{D}(Y_s)$. Since all the intersection numbers are integers, this is well defined. We omit the proof that this is the integral affine manifold B.

In the case of principally polarized abelian varieties, we have $\mathrm{Pic}(\mathcal{Y}) \cong \mathrm{NS}(Y_t)$ for any minimal model $\mathcal{Y}$. Furthermore, $Y = X$, each maximal dimensional cone $C^Y(\mathscr{T})$ is the cone of quadratic forms whose paving is coarser than $\mathscr{T}$. By ([AN99] Lemma 1.8), the paving $\mathscr{T}$ is the Delaunay decomposition of the corresponding quadratic form with respect to the lattice Y. The collection $\{C^Y(\mathscr{T})\}$ with their faces is the second Voronoi fan of $\mathcal{C}(X)^{\mathrm{rc}}$ with respect to the lattice $Y = X$. For the convenience of the reader, we recall the definition of the second Voronoi fan ([Nam80] Definition 9.8, Theorem 9.9).

Definition 3.49 (the second Voronoi fan)**.** Let $\mathcal{C}(X)^{\mathrm{rc}}$ be the rational closure of the cone of positive definite quadratic forms $\mathcal{C}(X)$. For any paving $\mathscr{P}$ of $X_{\mathbf{R}}$, set

$$\sigma(\mathscr{P}) := \{Q \in \mathcal{C}(X)^{\mathrm{rc}}; \text{the Delaunay decomposition of } Q \text{ with respect to } X \text{ is } \mathscr{P}\}.$$

The collection of cones $\{\sigma(\mathscr{P})\}$ is a fan, and is called *the second Voronoi fan* with respect to X.

We have proved

Theorem 3.50. *Let $\mathcal{Y}^{0,*}/\Delta^*$ be the mirror family for the principally polarized abelian varieties. For any relative minimal model $\mathcal{Y}/\Delta$, we have $\overline{\mathrm{Eff}}(\mathcal{Y}) = \overline{\mathcal{C}}(X)$. Moreover, the Mori fan of $\mathcal{Y}$ agrees with the second Voronoi fan.*

In general, $\overline{\mathrm{Eff}}(\mathcal{Y})$ is much bigger than $\overline{\mathcal{C}}(X)$. We need a section to get a canonical fan on $\mathcal{C}(X)^{\mathrm{rc}}$.

Proposition 3.51. *For any irreducible component D_α (for $\alpha \in I$) of Y_s, the complement $Y_s \backslash D_\alpha$ is contractible in $\mathcal{Y}$, and the contraction is denoted by $p_\alpha : \mathcal{Y} \dashrightarrow \mathcal{Y}_\alpha$. We call $\mathcal{Y}_\alpha$ a cusp model. It is not unique, but they are all isomorphic up to codimension 1. They are $\mathbf{Q}$-factorial, normal, and Gorenstein.*

Proof. We construct the cusp model $\mathcal{Y}_\alpha$ directly by the Mumford's construction. Fix α, consider the lattice $\alpha + \check{Y}$ in U^*. Construct a fan from the cones over a Delaunay decomposition $\mathscr{P}_D$ with respect to $\alpha + \check{Y}$. Take a Y-quasiperiodic, convex, piecewise affine function φ, we can get a relative complete model as before. By Mumford's construction, we get one of the models $\mathcal{Y}_\alpha$. □

By a similar argument as Proposition 3.48, the Mori fan of $\mathcal{Y}_\alpha$ is identified with the second Voronoi fan with respect to the lattice $\alpha+\check{Y}$. Furthermore, since the central fiber is irreducible, the restriction map $\mathrm{Pic}(\mathcal{Y}_\alpha) \to \mathrm{NS}(Y_t)$ is an isomorphism. The Mori fans are identified in $\mathrm{NS}(Y_t)$ because the second Voronoi fan doesn't depend on the translation of the lattice on U^*.

Now fix a cusp α. Consider the common Mori fan in $\mathrm{Pic}(\mathcal{Y}_\alpha)$. Each cone is an ample cone $\mathcal{K}(\mathcal{Y}'_\alpha)$ for some cusp model $\mathcal{Y}'_\alpha$. Define a section on this cone

$$\begin{aligned} \sigma_\alpha : \mathcal{K}(\mathcal{Y}'_\alpha) &\to \mathrm{Pic}(\mathcal{Y}) \\ \psi &\mapsto (p_\alpha)^*(\psi). \end{aligned}$$

That means, on each ample cone, we just pull back the Cartier divisor. σ_α is not linear, but piecewise linear and convex. Regard the function σ_α as a piecewise linear section $\mathrm{NS}(Y_t) \to \mathrm{Pic}(\mathcal{Y})$. Take the average on $\mathrm{NS}(Y_t)$, and define

$$\sigma = \frac{1}{d}\sum_\alpha \sigma_\alpha. \tag{3.17}$$

Proposition 3.52. *The section σ is linear.*

Proof. Consider the bending parameters of σ. Since all different linear pieces of σ are sections of r, all bending parameters live in the kernel $\ker(r) = \mathbf{Z}^d/\mathbf{Z}$. On the other hand, X is acting on X by translation. It induces an action of $X/\check{Y}$ on $\mathrm{Pic}(\mathcal{Y})$ since elements of $\mathrm{Pic}(\mathcal{Y})$ are functions on X. The set $\{\sigma_\alpha\}$ is a torsor under the action of $X/\check{Y}$. It follows that the image of σ is invariant under $X/\check{Y}$. Thus the bending parameters are also $X/\check{Y}$-invariant.

This implies that all the coefficients are the same, and the bending parameters are 0 in $\mathbf{Z}^d/\mathbf{Z}$. $\square$

Since σ is a linear section, it is easy to compute that $\sigma : \mathrm{NS}(Y_t) \to \mathrm{Pic}(\mathcal{Y})$ is

$$\sigma : Q \mapsto \varphi = \text{affine interpolation of } A_X|_X.$$

The image of σ is characterized by being invariant under the action of $X/\check{Y}$. We denote it by $\mathrm{Pic}^X(\mathcal{Y})$.

Remark 3.53. This choice of degeneration data for higher polarizations has been considered in [ABH02] as well as in [Nak10].

Identify $\mathrm{Pic}^X(\mathcal{Y})$ with the subspace of X-quasiperiodic functions in $\mathrm{Pic}(\mathcal{Y})$, i.e., the sections are the pull-backs from U^*/X. Consider the Mori fan of $\mathrm{Pic}(\mathcal{Y})$, and pull it back through σ. Each cone is of the form $\sigma^{-1}(C(\mathscr{P})) = \mathrm{Pic}^X(\mathcal{Y}) \cap C^Y(\mathscr{P})$, and is denoted by $C(\mathscr{P})$. Since every element in $C(\mathscr{P})$ is X-quasiperiodic, it can be identified with the associated quadratic form. Therefore, each $C(\mathscr{P})$ is the cone of quadratic forms whose paving is coarser than $\mathscr{P}$, and the pull back fan is the second Voronoi fan with respect to X for $\mathcal{C}(X)$. This is the fan we are going to use for the toroidal compactification, and it is denoted by $\Sigma(X)$.

3.2.2 Non-splitting Boundary Components

Now we consider the general case. Assume $\phi : Y \to X$ is of type $\mathfrak{d}$, the mirror family is $Y_t = E_1 \times E_2 \times \ldots \times E_g$, where $E_i = \mathbf{C}/(d_i^{-1}\mathbf{Z} + t\mathbf{Z})$. There is no Lagrangian section for the fiberation $X \to B$, but only local Lagrangian sections. Assume the maximal corank rational boundary component is associated to a maximal isotropic subspace $U \subset V$. Let $X^* := U \cap \Lambda$ be generated by $\{v_1, v_2, \dots, v_g\}$, and extend it to a basis $\{u_1', u_2', \dots, u_g', v_1, v_2, \dots, v_g\}$ of Λ. Under this new basis, E is

$$E = \begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix}.$$

Assume further the transformation matrix from the standard basis $\{\lambda_1, \dots, \lambda_g, \mu_1,$
$\dots, \mu_g\}$ of Λ to $\{u_1', u_2', \dots, u_g', v_1, v_2, \dots, v_g\}$ is M'^{-1}. Let

$$R^{-1} = \begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} M'^{-1},$$

where

$$A^{-1}B\mathfrak{d} = \frac{1}{2}S \tag{3.18}$$

$$A = \delta\mathfrak{d}^{-1} \tag{3.19}$$

R^{-1} maps $\{\lambda_1, \lambda_2, \dots, \lambda_g, \mu_1, \mu_2, \dots, \mu_g\}$ to $\{u_1, u_2, \dots, u_g, v_1, v_2, \dots, v_g\}$. Let $\widetilde{L}'$ be the real subspace of V generated by $\{u_1', u_2', \dots, u_g'\}$, and $\widetilde{L}$ be the subspace generated by $\{u_1, u_2, \dots, u_g\}$. The image of $\widetilde{L}'$ and $\widetilde{L}$ in the torus X are denoted by L' and L separately. L' is not a Lagrangian, but $(\widetilde{L}' \cap \Lambda) \oplus X^* = \Lambda$. Therefore we identify Y with $\widetilde{L}' \cap \Lambda$, and B with L'. On the other hand, L is a Lagrangian, but the map $f|_L : L \to B$ is not an isomorphism, only an unramified cover, with degree equal to the degree of the subgroup $\widetilde{L} \cap \Lambda \oplus X^*$ inside Λ. Use the splitting $X = B \oplus U/X^*$. On the universal cover, $\widetilde{L}$ is the graph of the function $x \mapsto \frac{1}{2}xS\mathfrak{d}^{-1}$, where x is written as a row vector.

Let $\{U_i\}$ be an open cover of B such that all U_i and $U_i \cap U_j$ are contractible. Choose local sections $\sigma_i : U_i \to L$ of f. Construct the Poincare bundles $\mathcal{P}_i$ over $f^{-1}(U_i) \times_{U_i} \check{f}^{-1}(U_i)$, then for any open subset $W \subset U_i$ we can define the local Fourier transform Four_i^W. Over $U_{ij} = U_i \cap U_j$, define a holomorphic line bundle P_{ij} on $\check{f}^{-1}(U_{ij}) \subset Y_t$.

$$P_{ij} = \mathrm{Four}_i^{U_{ij}}(\sigma_j(U_{ij}))$$

There are canonical isomorphisms $\alpha_{ijk} : P_{ij} \otimes P_{jk} \to P_{ik}$ over $\check{f}^{-1}(U_i \cap U_j \cap U_k)$, which are compatible over quadruple intersections. Therefore the collection (P_{ij}, α_{ijk}) defines a grebe e over Y_t. It is easy to compute the line bundles P_{ij}. Consider the sections σ_i, σ_j as lifts to $\widetilde{L}$. If the difference of $\sigma_j - \sigma_i$ over U_{ij} is equal to a translation by $v \in V$ such that the image of v in V/U is $\lambda_{ij} \in Y$, then (P_{ij}, ∇_{ij}) is a line bundle with the connection

$$\nabla_{ij} := \mathrm{d} + \pi i \langle \mathrm{d}\check{y}, \lambda_{ij} S\mathfrak{d}^{-1} \rangle.$$

The $0,1$-part of the unitary flat connection ∇_{ij} is integrable, and P_{ij} is a holomorphic line bundle.

Therefore, the difference between the general case and the splitting case is, when we apply the Fourier transform for $\check{\phi}$, we get sheaves twisted by the gerbe e. However, we don't really use the fact that the mirror map is obtained from the Fourier transform, and on the tropical level, everything goes the same as the splitting case. We can identify $\overline{\mathcal{C}}(X)$ with $\overline{\mathrm{Eff}}(Y_t)$. Construct the relative minimal models $\mathcal{Y}$, and we have the same exact sequence (3.16). Taking the average of the sections, we get a linear section $\sigma : \mathcal{K}(Y_t) \to \mathrm{Pic}(\mathcal{Y})$. Pull back the Mori fan along σ, we get the second Voronoi fan $\Sigma(X)$ with respect to X on $\mathcal{C}(X)$.

In sum, if F is a 0-cusp, we get a canonical fan supported on $\mathcal{C}(X)^{\mathrm{rc}}$, which is the second Voronoi fan with respect to the lattice X. If F_ξ is an arbitrary cusp, and $F_\xi \succ F$ for F a 0-cusp, we need $\Sigma(F_\xi) = \Sigma(F) \cap \mathcal{P}'(F)$ because of c) in Definition 3.12. Therefore, $\Sigma(F)_\xi$ is the second Voronoi fan for $X_{\xi,\mathbf{R}}$ with respect to X_ξ[5].

Definition 3.54. For each cusp F, we define the fan $\Sigma(F)$ (or $\Sigma(X)$) to be second Voronoi fan with respect to X (Definition 3.49), supported on $\mathcal{C}(X)^{\mathrm{rc}}$. Define $\widetilde{\Sigma} := \{\Sigma(F)\}$.

3.3 The Toroidal Compactification: The Construction

Proposition 3.55. *The collection of fans $\widetilde{\Sigma} = \{\Sigma(F)\}$ in Definition 3.54 is an admissible collection.*

Proof. First, we prove that $\Sigma(F)$ is an admissible fan for every F. Recall that

[5]For the proof, see part c) of the proof of Theorem 3.55 below.

$\Sigma(F)$ is an admissible fan with respect to $\mathrm{GL}(X)$ by [Nam80] Theorem 9.9. That means:

a') $\bigcup_{\sigma\in\Sigma} = \mathcal{C}(F)^{rc}$.

b') $M(\sigma)\in\Sigma$ for every $\sigma\in\Sigma$ and every $M\in\mathrm{GL}(X)$.

c') There are only finitely many orbits of simplices in Σ under the action of $\mathrm{GL}(X)$ on Σ, i.e., $\mathrm{GL}(X)\backslash\Sigma$ is a finite set.

Therefore, a) is satisfied. Since $\overline{P}(F)\subset\mathrm{GL}(X,Y)\subset\mathrm{GL}(X)$, b) is satisfied. Finally, $\overline{P}\subset\mathrm{GL}(X,Y)$ is of finite index by Proposition 3.31, and $\mathrm{GL}(X,Y)\subset\mathrm{GL}(X)$ is of finite index, then $\overline{P}(F)\subset\mathrm{GL}(X)$ is of finite index. c) is satisfied.

For b) in Definition 3.12. If $M\in\Gamma(\delta)$, $M:\Gamma^2U\to\Gamma^2U'$ is induced from the isomorphism $M^{-1}|_U:U\to U'$. And $M(\Lambda\cap U)=\Lambda\cap U'$. So the dual map $(M^{-1}|_U)^*:X'_{\mathbf{R}}\to X_{\mathbf{R}}$ maps X onto X'. Therefore $M:\mathcal{C}(X)\to\mathcal{C}(X')$ maps the second Voronoi fan $\Sigma(X)$ to $\Sigma(X')$.

For c) in Definition 3.12. If $F'\succ F$, then $U'\subset U$ respects the lattices. Therefore the induced quotient map $q_{\mathbf{R}}:X_{\mathbf{R}}\to X'_{\mathbf{R}}$ is from a quotient map of the lattices $q:X\to X'$. The pull back $\Gamma^2U'\to\Gamma^2U$ identifies $\mathcal{C}(X')$ with the positive semi-definite quadratic forms with nullspaces $\ker(q_{\mathbf{R}})$. Let $\mathscr{P}_{D'}$ be a Delaunay decomposition of $X'_{\mathbf{R}}$ and $\sigma'\in\Sigma(F')$ be the cone associated to $\mathscr{P}_{D'}$. The pull back $\mathscr{P}=q_{\mathbf{R}}^{-1}(\mathscr{P}_{D'})$ is a Delaunay decomposition of $X_{\mathbf{R}}$. (The cells are infinite in the direction of $\ker(q_{\mathbf{R}})$.) The cone σ associated to

$\mathscr{P}$ is a cone in $\Sigma(F)$, and it contains σ'. On the other hand, the supports of $\Sigma(F')$ and $\Sigma(F) \cap \Gamma^2 U'$ are the same. Therefore $\Sigma(F') = \Sigma(F) \cap \Gamma^2 U'$. □

Theorem 3.56. *Over* $\mathbf{C}$*, we have a toroidal compactification* $\overline{\mathcal{A}}_\Sigma$ *of* $\mathcal{A}_{g,\delta}$*. Furthermore,* $\overline{\mathcal{A}}_\Sigma$ *is projective.*

Proof. Apply Tai's criterion ([FC90] Definition 2.4) or ([AMRT10] Chap. IV Definition 2.1, Corollary 2.3). For each cone $\mathcal{C}(F)$, the fan $\Sigma(F)$ is the second Voronoi fan with respect to X, we can use the polarization function provided by $\Sigma(X_{\mathbf{R}}/X)$ in [Ale02]. □

Following [FC90], we are going to construct an arithmetic toroidal compactification from the admissible collection of fans $\widetilde{\Sigma}$. Since over $\mathbf{Z}$, the stack $\mathscr{A}_{g,\delta}$ is not a connected component of $\mathscr{A}_{g,d}$, in order to avoid the reduction of bad prime, we should work over $k = \mathbf{Z}[1/d]$. However, for some technical reasons, we have to work over $k = \mathbf{Z}[1/d, \zeta_M]$.

Chapter 4

The Compactification of the Moduli of Polarized Abelian Varieties

4.1 AN Families

4.1.1 Constructions

For this section, $k = \mathbf{Z}$. Recall the modified Mumford's construction in [AN99]. Fix a base scheme $S = \operatorname{Spec} R$, for R a Noetherian, excellent, normal integral domain, complete with respect to an ideal $I = \sqrt{I}$. Denote the residue ring R/I by κ, and the field of fractions by K. The closed subscheme is denoted by $S_0 = \operatorname{Spec} \kappa$, and the generic point by $\eta = \operatorname{Spec} K$. Recall the categories DEG and DD from [FC90]. The objects of the category $\mathrm{DEG}_{\mathrm{ample}}$, are pairs $(G, \mathcal{L})$, where G is semi-abelian over S, with G_η abelian over K, and $\mathcal{L}$ is an invertible sheaf on G, with $\mathcal{L}_\eta$ ample. The morphisms are group morphisms that respect the invertible sheaves. An object of $\mathrm{DD}_{\mathrm{ample}}$ is the following degeneration data:

1. An abelian scheme A/S of relative dimension g', a split torus T/S defined by the character group $X \cong \mathbf{Z}^r$, with $r = g - g'$, and a semi-abelian group

scheme $\widetilde{G}$ defined by $c : X \to A^t$. We use the same letter to denote the group X, and the correspondent constant sheaf X on S.

$$1 \longrightarrow T \longrightarrow \widetilde{G} \xrightarrow{\pi} A \longrightarrow 0.$$

2. A rank r free abelian group $Y \cong \mathbf{Z}^r$ and the constant sheaf Y over S.

3. A homomorphism between group schemes $c^t : Y \to A$. This is equivalent to an extension

$$1 \longrightarrow T^t \longrightarrow \widetilde{G}^t \xrightarrow{\pi^t} A^t \longrightarrow 0.$$

4. An injection $\phi : Y \to X$ of type $\mathfrak{d}_1$.

5. A homomorphism $\iota : Y_\eta \to \widetilde{G}_\eta$ over S_η lying over c^t_η. This is equivalent to a trivialization of the biextension $\tau : 1_{Y\times X} \to (c^t \times c)^*\mathcal{P}^{-1}_{A,\eta}$. Here $\mathcal{P}_A$ is the Poincaré sheaf on $A\times A^t$ which has a canonical biextension structure. We require that the induced trivialization $\tau \circ (\mathrm{Id} \times \phi)$ of $(c^t \times c\circ\phi)^*\mathcal{P}^{-1}_{A,\eta}$ over $Y \times Y$ is symmetric. The trivialization τ is required to satisfy the following positivity condition: $\tau(\lambda, \phi(\lambda))$ for all λ extends to a section of $\mathcal{P}^{-1}_A$ on $A \times_S A^t$, and is 0 modulo I if $\lambda \neq 0$.

6. An ample sheaf $\mathcal{M}$ on A inducing a polarization $\lambda_A : A \to A^t$ of type δ' such that $\lambda_A c^t = c\phi$. This is equivalent to giving a T-linearized sheaf $\widetilde{\mathcal{L}} = \pi^*\mathcal{M}$ on $\widetilde{G}$.

7. An action of Y on $\widetilde{\mathcal{L}}_\eta$ compatible with ϕ. This is equivalent to a cubical trivialization $\psi : 1_Y \to (c^t)^*\mathcal{M}^{-1}_\eta$, which is compatible with the trivialization $\tau \circ (\mathrm{Id} \times \phi)$.

Here ψ being compatible with τ means as follows. First, for an integer n and a subset $I \subset \{1, \dots, n\}$, let $m_I : A^n \to A$ be the morphism that adds the i-th coordinates for all $i \in I$. If I empty, m_I is the constant morphism to the identity section of A. For any sheaf $\mathcal{M}$ over an abelian scheme A, define a sheaf $\Lambda(\mathcal{M})$ over the product $A \times A$ by

$$\Lambda(\mathcal{M}) := \bigoplus_{I \subset \{1,2\}} m_I^* \mathcal{M}^{(-1)^{\# I}}.$$

By the theorem of cube, if A is an abelian scheme, and $\mathcal{M}$ defines a polarization λ_A, then $\Lambda(\mathcal{M})$ is a symmetric biextension and agrees with $(\mathrm{Id} \times \lambda_A)^* \mathcal{P}_A$. Therefore $(c^t \times c^t)^* \Lambda(\mathcal{M})^{-1} = (c^t \times c \circ \phi)^* \mathcal{P}_A^{-1}$. We say ψ and τ are *compatible* if $\Lambda(\psi) = \tau \circ (\mathrm{Id} \times \phi)$ as trivializations of the symmetric biextension over $Y \times Y$.

Following [AN99], we denote the rigidified line bundle $c(\alpha)$ by $\mathcal{O}_\alpha$, and the rigidified line bundle $\mathcal{M} \otimes \mathcal{O}_\alpha$ by $\mathcal{M}_\alpha$. For any S-point $a : S \to A$, and any line bundle $\mathcal{L}$ on A, the pull back $a^* \mathcal{L}$ is denoted by $\mathcal{L}(a)$. For any $\lambda \in Y$, $\alpha \in X$, $\tau(\lambda, \alpha) \in \mathcal{O}_\alpha(c^t(\lambda))_\eta^{-1} = \mathcal{P}_A^{-1}(c^t(\lambda), c(\alpha))_\eta$, and $\psi(\lambda) \in \mathcal{M}(c^t(\lambda))_\eta^{-1}$ are K-sections. Therefore the data τ and ψ gives trivializations $\psi(\lambda)^d \tau(\lambda, \alpha) : (\mathcal{M}^d \otimes \mathcal{O}_\alpha)_\eta^{-1}(c^t(\lambda)) \cong \mathcal{O}_K$ for every $\lambda \in Y$, $\alpha \in X$. By ([Ols08] Lemma 5.2.2.), the data $\psi(\lambda)^d \tau(\lambda, \alpha)$ is equivalent to an isomorphism of line bundles

$$\psi(\lambda)^d \tau(\lambda, \alpha) : T_{c^t(\lambda)}^* (\mathcal{M}^d \otimes \mathcal{O}_\alpha)_\eta \to \mathcal{M}^d \otimes \mathcal{O}_{\alpha + d\phi(\lambda), \eta}.$$

By the property of biextensions, ([Ols08] Proposition 2.2.13), it defines an action of the group Y on the big graded algebra

$$\mathcal{S} = \left(\prod_{d\geqslant 0}\big(\bigoplus_{\alpha\in X}\mathcal{O}_\alpha\big)\otimes\mathcal{M}^d\theta^d\right)\otimes_R K,$$

denoted by S^*_λ.

According to [AN99], in order to construct a proper family of SQAV, we have to choose embeddings

$$\mathcal{M}_\alpha \to \mathcal{M}_{\alpha,\eta}$$

for every representative $\alpha \in X/\phi(Y)$.

In order to construct the pair SQAP, we have to choose additional data

$$\vartheta_{A,\alpha} \in \mathrm{H}^0(A, \mathcal{M}_\alpha)$$

for every representative $\alpha \in X/\phi(Y)$.

Now assume P is a toric monoid, $P = \sigma_P^\vee \cap P^{gp}$, and $\alpha : P \to R$ is a prelog structure which maps the toric maximal ideal $P\backslash\{0\}$ to I. Fix data: an integral, Y-quasiperiodic, P-convex piecewise affine function $\varphi : X_{\mathbf{R}} \to P^{\mathrm{gp}}_{\mathbf{R}}$ decided by a collection of bending parameters $\{p_\rho \in P\backslash\{0\}\}$. For simplicity, the composition $\alpha\circ\varphi : X_{\mathbf{R}} \to R$ is also denoted by φ. Suppose the associated quadratic form Q is positive definite, i.e. $Q(x) \in \sigma_P^\vee\backslash\{0\}$ for $x \neq 0$, and the associated decomposition $\mathscr{P}$ is bounded. Construct the following function

$$a_t : Y \to \mathbb{G}_m(K),$$
$$b_t : Y \times X \to \mathbb{G}_m(K),$$

by

$$a_t(y) := \mathrm{X}^{\varphi(\phi(y))}. \quad \forall y \in Y.$$

$\mathrm{X}^{\varphi(\phi(y))}$ is an element in R through the map $k[P] \to R$.

Define b_t by requiring that

$$\mathrm{X}^{\varphi(\alpha+\phi(y))} = a_t(y)b_t(y,\alpha)\mathrm{X}^{\varphi(\alpha)}.$$

By the properties of φ, a_t is quadratic, b_t is bilinear and $b_t(y, \phi(x))$ is symmetric on $Y \times Y$. Recall that, by ([Ols08] 2.2.8), for any biextension over $Y \times X$, the automorphism is classified by $\mathrm{Hom}(Y \otimes X, \mathbb{G}_m)$. If we require the extension is symmetric on $Y \times \phi(Y)$, then the bilinear form $b \in \mathrm{Hom}(Y \otimes X, \mathbb{G}_m)$ should satisfy that $b(y, \phi(x))$ be symmetric. Furthermore, the automorphism of a central extension over Y is classified by quadratic functions over Y. Fix a biextension (resp, central extension), regard the set of trivializations as a torsor over $\mathrm{Hom}(Y \times X, \mathbb{G}_m)$ (resp, quadratic forms). For any trivialization τ of a biextension, (reps, any trivialization ψ of a central extension), denote the trivialization obtained by the action of b_t^{-1} (resp, a_t^{-1}) by $b_t^{-1}\tau$ (resp, $a_t^{-1}\psi$).

Definition 4.1. Fix the data b_t, a_t obtained from φ. We say the combinatorial data φ is *compatible with the trivializations τ and ψ* if $b_t^{-1}\tau$ can be extended to a trivialization of the biextension $(c^t \times c)^*\mathcal{P}_A^{-1}$ over S, and $a_t^{-1}\psi$ can be extended to a trivialization of the central extension $(c^t)^*\mathcal{M}^{-1}$ over S.

The data φ gives the choice of embeddings $\mathcal{M}_x \to \mathcal{M}_{x,\eta}$, therefore we can use constructions in [AN99] to construct the family of varieties over S. More explicitly, consider the piecewise linear function $\tilde{\varphi} : \mathbb{X}_{\mathbf{R}} \to P^{gp}_{\mathbf{R}}$ and its graph in $\mathbb{X}_{\mathbf{R}} \times P^{gp}_{\mathbf{R}}$. The piecewise linear function $\tilde{\varphi}$ is integral because the bending parameters p_ρ are all in P. Denote the convex polytope P-above the graph by $Q_{\tilde{\varphi}}$. Consider the graded $\mathcal{O}_A$-algebra

$$\mathcal{R} := \bigoplus_{((d,\alpha),p)\in Q_{\tilde{\varphi}}} \mathrm{X}^p \otimes \mathcal{O}_\alpha \otimes \mathcal{M}^d\theta^d.$$

Since φ and τ, ψ are compatible, the action $S^*_\lambda, (\lambda \in Y)$ is

$$\psi(\lambda)^d\tau(\lambda,\alpha) : T^*_{c^t(\lambda)}(\mathrm{X}^p \otimes \mathcal{O}_\alpha \otimes \mathcal{M}^d\theta^d) = \psi'(\lambda)^d\tau'(\lambda,\alpha)\mathrm{X}^p a_t(\lambda)^d b_t(\lambda,\alpha) \otimes \mathcal{O}_{\alpha+d\phi(\lambda)} \otimes \mathcal{M}^d\theta^d, \tag{4.1}$$

for $\psi'(\lambda)^d\tau'(\lambda,\alpha)$ R-sections of $\mathcal{M}^d \otimes \mathcal{O}_\alpha(c^t(\lambda))$. Therefore, the actions S^*_λ preserve the subalgebra $\mathcal{R}$. Let $\widetilde{\mathcal{X}}$ denote the scheme $\mathbf{Proj}\,\mathcal{R}$. Construct the infinite toric variety X_φ from the convex polytope $Q_{\tilde{\varphi}}$. This is a toric degeneration over the toric variety $\mathrm{Spec}\, k[P]$. Denote the pull back of X_φ along $k[P] \to R$ by $\widetilde{P}^r$. As in ([AN99], 3B, 3.22), the total space $\widetilde{\mathcal{X}}$ is isomorphic to contracted product $\widetilde{P}^r \times^T \widetilde{G}$ over S.

Lemma 4.2. *Every irreducible component of $\widetilde{X}_0$ is reduced, and proper over S_0.*

Proof. Use the isomorphism $\widetilde{\mathcal{X}} \cong \widetilde{P}^r \times^T \widetilde{G}$ over S. The total space $\widetilde{\mathcal{X}}$ is covered by $U(\omega) \times^T \widetilde{G}$, where $U(\omega)$ corresponds to the affine toric variety for the vertex ω of $Q_{\tilde{\varphi}}$. Each vertex ω of $Q_{\tilde{\varphi}}$ corresponds to its image $\omega \in \mathscr{P}$. The action S_y^* maps $U(\omega) \times^T \widetilde{G}$ to $U(\omega + \phi(y)) \times^T \widetilde{G}$. Reduce to k, $\widetilde{X}_0$ is the contracted product of $\widetilde{P}_0^r := \widetilde{P}^r \times_{k[P]} S_0$ with $\widetilde{G}_0$. Since $S_0 = \operatorname{Spec} R/I$, $P \backslash \{0\}$ is mapped into I, and φ is integral, the scheme $\widetilde{P}_0^r$ is $X_{\mathscr{T}}$ as defined in Chap. 2 Sect. 2.1.2. Each irreducible component of $\widetilde{P}_0^r$ is the toric variety X_σ associated to the maximal cell $\sigma \in \mathscr{T}$. As a result, each irreducible component of $\widetilde{X}_0$ is a fiber bundle over A_0 with the fiber X_σ, and is thus reduced and proper over S_0. $\square$

Lemma 4.3. *The data $(\widetilde{\mathcal{X}}, \mathcal{O}(1), S_\lambda^*, \widetilde{G})$ is a relative complete model as defined in ([FC90], III. Definition 3.1).*

Proof. The proof is similar to ([AN99], Lemma 3.24). It is not necessary to check the complete condition, since we have shown that every irreducible component of $\widetilde{X}_0$ is proper over S_0. To construct an embedding $\widetilde{G} \to \widetilde{\mathcal{X}}$, consider the dual fan $\Sigma(\tilde{\varphi})$ of $Q_{\tilde{\varphi}}$ in $\mathbb{X}_{\mathbf{R}}^* \times (P_{\mathbf{R}}^{gp})^*$. It defines a toric degeneration $\widetilde{P}^r$ over the affine toric variety $\operatorname{Spec} k[P]$. It follows that we can lift the fan σ_P as a subfan in $\Sigma(\tilde{\varphi})$. Since σ_P is a fan defined by faces of one rational polyhedral cone, the subfan induces an embedding of the trivial torus bundle T into $\widetilde{P}^r$. By ([AN99], B 3.22), the embedding $T \to \widetilde{P}^r$ induces the embedding $\widetilde{G} \cong T \times^T \widetilde{G} \to \widetilde{P}^r \times^T \widetilde{G} \cong \widetilde{\mathcal{X}}$. $\square$

As in [AN99], reduce to I^n for various $n \in \mathbf{N}$, and take the quotient by Y, then by Grothendieck's existence theorem ([GD63] EGA III_1 5.4.5), we get an algebraic family of pairs $(\mathcal{X}, \mathcal{L})$ over the I-complete base S. Denote the projective morphism by $\pi : \mathcal{X} \to S$.

4.1.2 Convergence

Notice that in the construction above, the only place we use the condition that $P\backslash\{0\}$ is mapped to I is in the proof of Lemma 4.2. In this case, closed subscheme S_0 is sent to the 0-strata in the affine toric variety $\operatorname{Spec} k[P]$. However, we may also consider other closed strata.

Suppose $P = \sigma_P^\vee \cap P^{\mathrm{gp}}$ is a toric monoid obtained from a rational polyhedral cone σ_P, and $\varphi : X_{\mathbf{R}} \to P_{\mathbf{R}}^{\mathrm{gp}}$ is a function as above. Let $\mathfrak{m}$ be the maximal toric ideal $P\backslash\{0\}$, J be a prime toric ideal of P, and F be the face $P\backslash\, J$. Assume $F = \tau^\perp \cap \sigma_P^\vee$ for τ a face of σ_P. Consider the monoid $P' = P/F$ and the piecewise affine function φ' defined as the composition of φ with $P_{\mathbf{R}}^{\mathrm{gp}} \to (P')_{\mathbf{R}}^{\mathrm{gp}}$. The associated paving $\mathscr{P}'$ of φ' is coarser than the associated paving $\mathscr{P}$ of φ. Suppose that $\mathscr{P}'$ is still a bounded decomposition. Let P_F be the localization of P with respect to the face F, and J_F be the toric ideal of P_F generated by J.

Definition 4.4. All the notations as above. We call the data $(P', \varphi', \mathscr{P}')$, the data associated to the face F.

Lemma 4.5. *Every irreducible component of $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P]/J$ is proper* *over* $\operatorname{Spec} k[P]/J$.

Proof. The family $\widetilde{P}^r \to \operatorname{Spec} k[P]$ is an infinite toric degeneration constructed from the infinite piecewise affine function φ. The restriction to $\operatorname{Spec} k[P]/J$ is still a toric degeneration. Since $\operatorname{Spec} k[P_F]/J_F$ is open and dense in $\operatorname{Spec} k[P]/J$, and the closure of an irreducible space is irreducible, each irreducible component of $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P]/J$ is a toric degeneration of an irreducible component of $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P_F]/J_F$. Therefore, it suffices to prove the statement for $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P_F]/J_F$.

Since P is toric monoid from a rational polyhedral cone $\sigma_P^\vee$, we can choose a splitting $P_F = P' \times F^{\mathrm{gp}}$. Consider the function $\tilde{\varphi}' : \mathbb{X}_{\mathbf{R}} \to (P')^{\mathrm{gp}}_{\mathbf{R}}$ and the convex polytope $Q_{\tilde{\varphi}'}$ over the graph. Let $\widetilde{P}'$ be the toric variety assoicated to the convex polytope $Q_{\tilde{\varphi}'}$. The localization $\widetilde{P}^r_F := \widetilde{P}^r \times_{k[P]} \operatorname{Spec} k[P_F]$ is isomorphic to $\widetilde{P}' \times \operatorname{Spec} k[F^{\mathrm{gp}}]$ over $\operatorname{Spec} k[P'] \times \operatorname{Spec} k[F^{\mathrm{gp}}]$. Since J corresponds to the maximal ideal $\mathfrak{m}' = P' \backslash \{0\}$, it reduces to the case $\widetilde{P}' \times_{k[P']} \operatorname{Spec} k[P']/\mathfrak{m}'$. Again this is $X_{\mathscr{P}'}$, with each irreducible component the proper toric variety X_σ for a bounded polytope $\sigma \in \mathscr{P}'$. $\square$

Definition 4.6. For any prime toric ideal J, associate the face F, the quotient monoid P', the function $\varphi' : X_{\mathbf{R}} \to (P')^{\mathrm{gp}}_{\mathbf{R}}$, and the paving $\mathscr{P}'$ as above. A toric prime ideal J is called *admissible* for φ if the paving $\mathscr{P}'$ is bounded.

Lemma 4.7. *Given a toric ideal J such that $J = \cap_i J_i$ for each J_i an admissible prime toric ideal. Then every irreducible component of $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P]/J$ is proper over* $\operatorname{Spec} k[P]/J$.

Proof. The irreducible components of $\operatorname{Spec} k[P]/J$ are $\operatorname{Spec} k[P]/J_i$. Each irreducible component of $\widetilde{P}^r \times_{k[P]} \times \operatorname{Spec} k[P]/J$ is contained in the family over some $\operatorname{Spec} k[P]/J_i$ for some J_i. It reduces to Lemma 4.5. $\square$

The set of all admissible prime toric ideals for φ is finite. Take the intersection of all these admissible prime ideals, we get a toric ideal denoted by I_φ. The correspondent ideal in the ring $k[P]$ is also denoted by I_φ. Fix a Noetherian normal integral domain R, of finite type over a field or Dedekind domain k of characteristic 0, and a chart $\alpha : P \to R$. The induced ring homomorphism $k[P] \to R$ is also denoted by α.

Definition 4.8. Let F be a face of the toric monoid P, and $J = P \backslash F$ is the prime toric ideal. Let $\mathscr{P}'$ be the paving associated with the face F. A prime ideal $\mathfrak{p} \subset R$ is *in the interior of the $\mathscr{P}'$-strata* if J is the maximal prime toric ideal such that the correspondent ideal $J \subset \alpha^{-1}(\mathfrak{p})$.

Definition 4.9. An ideal $I \subset R$ is called *admissible* for φ if its radical $\sqrt{I}$ contains $\alpha(I_\varphi)$. The set of admissible ideal is denoted by $\mathfrak{A}_\varphi$.

Remark 4.10. Both of the definitions only depend on the log structure decided by the chart $\alpha : P \to R$, not the choice of the chart, because the ideals $\alpha^{-1}(I)$ and $\alpha^{-1}(\mathfrak{p})$ do not depend on the choice of the chart α.

Lemma 4.11. *If $I \subset I'$ and $I \in \mathfrak{A}_\varphi$, then $I' \in \mathfrak{A}_\varphi$. If $\sqrt{I} \in \mathfrak{A}_\varphi$, then $I \in \mathfrak{A}_\varphi$. The collection $\mathfrak{A}_\varphi$ is closed under finite intersection.*

Proof. The first and the second statements follow directly from the definition. The third statement follows from the fact $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. $\square$

Let I_φ denote the ideal $I_\varphi \otimes_{k[P]} R$ of R. Fix an ideal $I \subset R$, admissible for φ. Take the I-adic completion $\widehat{R}_I$.

Lemma 4.12. *The complete ring $\widehat{R}_I$ is a Noetherian, excellent, normal, integral domain, provided that the closed subscheme* $\operatorname{Spec} R/I$ *is connected.*

Proof. Since R is excellent, the I-adic completion $\widehat{R}_I$ is regular over R. ([GD67] EGA IV_2 7.8.3 (v)). In particular, since R is normal, $\widehat{R}_I$ is normal, and $\widehat{R}_I$ is a product of normal integral domains. If $\operatorname{Spec} R/I$ is connected, the set of closed points is connected. $\widehat{R}_I$ is a normal integral domain. Moreover, $\widehat{R}_I$ remains excellent by ([Val76] Theorem 9). $\square$

Proposition 4.13. *Assume there is an object of* DD_{ample} *over R that is compatible with φ, and I is an admissible ideal for φ, then there exists a projective family $(\mathcal{X}, \mathcal{L})$ over $S = \operatorname{Spec} \widehat{R}_I$ such that the generic fiber $\mathcal{X}_\eta \cong \widetilde{G}_\eta$ is abelian with an ample polarization.*

Proof. Check the construction. The only thing we need to proof is a statement similar to Lemma 4.2. This follows from Lemma 4.7. $\square$

Lemma 4.14. *The total space $\mathcal{X}$ is irreducible.*

Proof. This is ([FC90] III. Proposition 4.11). $\square$

Remark 4.15. We call this construction AN construction because it is introduced in the paper [AN99] when the base is a complete discrete valuation ring.

4.1.3 Properities

First we point out the relation between AN construction and the standard construction in [Ols08]. Recall $S(X) = \mathbf{N} \oplus X$. Introduce the monoid $S(X) \rtimes P$ on the underlying set $S(X) \times P$ with the addition law

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + \tilde{\varphi}_{\mathscr{P}}(\alpha) + \tilde{\varphi}_{\mathscr{P}}(\beta) - \tilde{\varphi}_{\mathscr{P}}(\alpha + \beta)).$$

Just as in the GKZ case, $S(Q_{\varphi}) = Q_{\tilde{\varphi}} \cong S(X) \rtimes P$. We have

$$\widetilde{P}^r \cong \operatorname{Proj} k[S(X) \rtimes P] \otimes_{k[P]} R.$$

Under this isomorphism, the lowest elements $((d, \alpha), p)$ in $Q_{\tilde{\varphi}}$ are mapped to $((d, \alpha), 0)$ in $S(X) \rtimes P$. As a result $\mathcal{R}$ is isomorphic to

$$\bigoplus_{(d,\alpha) \in S(X)} \mathcal{O}_{\alpha} \otimes \mathcal{M}^d \theta^d$$

with multiplication the composition of $\tilde{\varphi}(m, \alpha) + \tilde{\varphi}(n, \beta) - \tilde{\varphi}((m, \alpha) + (n, \beta))$ and the canonical isomorphism

$$(\mathcal{O}_{m\alpha} \otimes \mathcal{M}^m) \otimes (\mathcal{O}_{n\beta} \otimes \mathcal{M}^n) \cong \mathcal{O}_{m\alpha + n\beta} \otimes \mathcal{M}^{m+n}.$$

Reduction to I^n for each n, this is the standard construction in ([Ols08] 5.2), except that we have a toric monoid P instead of $H_{\mathscr{P}}$. Under this isomorphism, the data τ, ψ in [Ols08] corresponds to $b_t^{-1}\tau$ and $a_t^{-1}\psi$ in our case. The constructions of group actions and log structures in [Ols08] for the standard constructions can be applied to our case directly.

From the degeneration data, by Mumford's original construction ([Mum72] pp. 297), or ([FC90] pp. 66), we can also construct a semi-abelian scheme G smooth over S ([FC90] Chap. III Proposition 4.10), such that the formal scheme along I, $\widehat{G} \cong (\widetilde{G})\widehat{}$.

Consider the reduction over I^n. We can define the action ϱ_n of $\widetilde{G}_n$ on $\mathcal{X}_n$ as in [Ols08] 4.1. In other words, we get a compatible system $(\mathcal{X}_n, \mathcal{L}_n, G_n, \varrho_n)$ over $S_n := \operatorname{Spec} R/I^n$.

The choice of $\mathcal{M}$ gives a T-linearization of the line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}$. The action of T gives the Fourier decomposition of $\mathrm{H}^0(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$. By the Heisenberg relation of the action of Y and $\widetilde{G}$, the decomposition is preserved by the action of Y. Therefore, we have the Fourier decomposition

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}) = \bigoplus_{\alpha \in X/\phi(Y)} \mathrm{H}^0(A, \mathcal{M}_\alpha). \tag{4.2}$$

Let $\mathscr{P}^0 \subset X/\phi(Y)$ be a set of representatives of the vertices of the paving $\mathscr{P}$. For every $\alpha \in \mathscr{P}^0$, choose $\vartheta_{A,\alpha} \in \mathrm{H}^0(A, \mathcal{M}_\alpha)$ such that the residue $\vartheta_{A,\alpha,s} \neq 0$ for any $s \in S$. Define a section $\vartheta_n \in \mathrm{H}^0(\mathcal{X}_n, \mathcal{L}_n)$ as a descent from

$$\tilde{\vartheta} = \sum_{\lambda \in Y} \sum_{\alpha \in \mathscr{P}^0} S_\lambda^*(\vartheta_{A,\alpha}). \tag{4.3}$$

This is a finite sum for every admissible ideal I^n. Let Θ_n denote the divisor defined by ϑ_n. The collection $\{\Theta_n\}$ gives a divisor Θ on $\mathcal{X}$ such that $(\mathcal{X}, \Theta)$ is a stable pair in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.

Lemma 4.16. *For any admissible ideal I, and any geometric point $\bar{x} \to \operatorname{Spec} R/I$, the geometric fiber $(X_{\bar{x}}, \mathcal{L}_{\bar{x}}, \varrho_{\bar{x}}, \Theta_{\bar{x}})$ is a stable semiabelic pair as defined in [Ale02]. In particular, if the image x is in the $\mathscr{P}'$-strata, for a bounded decomposition $\mathscr{P}'$ of $X_{\mathbf{R}}$, then $(X_{\bar{x}}, \mathcal{L}_{\bar{x}}, \varrho_{\bar{x}})$ is an element in $M^{fr}[\Delta_{\mathscr{P}'}, c, c^t, \mathcal{M}](\kappa(\bar{x}))$ defined by $\psi_0^{(\cdot)}(\cdot)\tau_0(\cdot,\cdot) \in Z^1(\Delta_{\mathscr{P}'}, \hat{\underline{\mathbb{X}}})$, and $(X_{\bar{x}}, \mathcal{L}_{\bar{x}}, \varrho_{\bar{x}}, \Theta_{\bar{x}})$ is an element of $MP^{fr}[\Delta_{\mathscr{P}'}, c, c^t, \mathcal{M}](\kappa(\bar{x}))$ defined by $(\tilde{\vartheta}_0, \psi_0^{(\cdot)}(\cdot)\tau_0(\cdot,\cdot)) \in Z^1(\Delta_{\mathscr{P}'}, \hat{\mathbb{M}}^*)$. Here $\kappa(\bar{x})$ is the residue field for $\bar{x}$, and $\Delta_{\mathscr{P}'}$ is the complex defined by the decomposition $\mathscr{P}'$ on $X_{\mathbf{R}}/\check{Y}$. The notations $\psi_0, \tau_0, \tilde{\vartheta}_0$ is defined in the proof. For other notations, see [Ale02].*

Proof. Let J and F be the prime toric ideal and the face associated to $\mathscr{P}'$. We can localize with respect to F first, and replace φ by φ', which gives the decomposition $\mathscr{P}'$. Consider the geometric fiber $\widetilde{X}_{\bar{x}}$. Since $\alpha(J) \subset \mathfrak{m}_x$, if (m, α) and (n, β) is not in the same cell in $\mathscr{P}'$, $\tilde{\varphi}'(m, \alpha) + \tilde{\varphi}'(n, \beta) - \tilde{\varphi}'(m + n, \alpha + \beta)$ is mapped to zero in $\kappa(\bar{x})$. Therefore, denote $\mathrm{X}^{(d,\alpha), \tilde{\varphi}'(d,\alpha)} \otimes \mathcal{O}_\alpha \otimes \mathcal{M}^d \theta^d$ by $\mathcal{M}_{d,\alpha}$, $(\widetilde{X}_{\bar{x}}, \widetilde{\mathcal{L}}_{\bar{x}})$ is the gluing of $(P, L)[\delta, c, \mathcal{M}]$ for each $\delta \in \mathscr{P}'$. Here $(P, L)[\delta, c, \mathcal{M}]$ is defined in ([Ale02] Definition 5.2.5). The action of Y is induced by $\psi^{(\cdot)}(\cdot)\tau(\cdot,\cdot)$, i.e. the residue of ψ', τ' in $\kappa(\bar{x})$ for ψ', τ' in Equation (4.1). Let the residues be denoted by ψ_0, τ_0. It is an element in $Z^1(\Delta_{\mathscr{P}'}, \widehat{\underline{\mathbb{X}}})$.

The residue of $\tilde{\vartheta}$ in $\kappa(\bar{x})$ is denoted by $\tilde{\vartheta}_0$. By the definition of ϑ_α, the residue $\tilde{\vartheta}_0$ is in $C^0(\Delta_{\mathscr{P}'}, \widehat{\underline{\mathrm{Fun}}_{\geqslant 0}})$, and $\Theta_{\bar{x}}$ does not contain any $G_{\bar{x}}$-strata entirely. The condition that $\tilde{\vartheta}$ is Y-invariant means that $(\tilde{\vartheta}_0, \psi_0^{(\cdot)}(\cdot)\tau_0(\cdot,\cdot))$ is an element in $Z^1(\Delta_{\mathscr{P}'}, \widehat{\mathbb{M}}^*)$. □

Lemma 4.17. *For any admissible ideal I and any n, the family $(\mathcal{X}_n, \mathcal{L}_n, \varrho_n, \Theta_n)$ over* $\operatorname{Spec} R/I^n$ *is an object in* $\overline{\mathscr{A}\mathscr{P}}_{g,d}(\operatorname{Spec} R/I^n)$.

Proof. By Lemma 4.16, it suffices to prove that $(\mathcal{X}_n, \mathcal{L}_n)$ is flat over $S_n := \operatorname{Spec} R/I^n$. It is a general property of the Mumford type constructions. First, $\widetilde{\mathcal{X}}$ is flat over A, since $\mathcal{R}$ is locally free as $\mathcal{O}_A$-module. Secondly, when we take the quotient by Y, we first take a quotient by Y_n, such that $Y_n \subset Y$ is of finite index, and the action of Y_n on $\widetilde{\mathcal{X}}_n$ is free. So this quotient preserves the flatness. Finally, the quotient by the finite group Y/Y_n. Consider the representation of the finite group Y/Y_n. The Y/Y_n-invariant part is a direct summand. Therefore $\mathcal{X}_n$ is flat over S_n. Since the line bundles $\mathcal{L}_n^m$ are flat over $\mathcal{O}_{\mathcal{X}_n}$, they are flat over S_n. □

Let the degree of the polarization be d. This defines an object in the inverse limit $\varprojlim \overline{\mathscr{A}\mathscr{P}}_{g,d}(S_n)$. By Grothendieck's existence theorem ([GD63] EGA III$_1$ 5.4.5), we get a family $(\mathcal{X}, \mathcal{L}, G, \varrho, \Theta)$ in $\overline{\mathscr{A}\mathscr{P}}_{g,d}(S)$.

Corollary 4.18. The coherent sheaves $\pi_*\mathcal{L}^m$ are locally free of rank dm^g for all $m \geqslant 0$. The coherent sheaves $R\pi_*^i\mathcal{L}^m = 0$ for all $m > 0, i > 0$. In particular, we have $\pi_*\mathcal{L}^m \otimes \kappa(x) \cong \mathrm{H}^0(X_x, \mathcal{L}_x^m)$.

Proof. For any $x \in S$, consider $\bar{x} \to S$ over the image x, with $\kappa(\overline{x})$ the algebraic closure of $\kappa(x)$. Consider the fiber $(X_{\bar{x}}, \mathcal{L}_{\bar{x}})$. By ([Ale02] Theorem 5.4.1), $H^i(X_{\bar{x}}, \mathcal{L}_{\bar{x}}^m) = 0$ for any $i, m \geqslant 1$. Thus $H^i(X_x, \mathcal{L}_x^m) = 0$ for any $i, m \geqslant 1$. Since the base S is still Noetherian, ([AM69] Theorem 10. 26), and

$\mathcal{L}^m$ is flat over S, we can apply ([Har77] Chap. III Theorem 12.11). We have $R\pi^i_*\mathcal{L}^m = 0$ for all $i, m > 0$, and $\pi_*\mathcal{L}^m \otimes \kappa(x) \to \mathrm{H}^0(X_x, \mathcal{L}^m_x)$ is surjective. Apply ([Har77] Chap. III Theorem 12.11) again, $\pi_*\mathcal{L}^m$ are locally free, and $\pi_*\mathcal{L}^m \otimes \kappa(x) \cong \mathrm{H}^0(X_x, \mathcal{L}^m_x)$. Since for the generic η, X_η is an abelian variety, and $\mathrm{H}^0(X_\eta, \mathcal{L}^m_\eta) = dm^g$, $\pi_*\mathcal{L}^m$ is of rank dm^g. □

By abusing of notations, the morphism $A \to S$ is also denoted by π. Then $\pi_*\mathcal{M}$ is also locally free. The Fourier decomposition is

$$\pi_*\mathcal{L} \cong \bigoplus_{\alpha \in X/\phi(Y)} \pi_*\mathcal{M}_\alpha. \tag{4.4}$$

For any coherent sheaf $\mathcal{F}$ over S, define $\mathcal{F}^*$ to be the subsheaf

$$\mathcal{F}^*(U) := \{f \in \mathcal{F} : f_x \notin \mathfrak{m}_x\mathcal{F}_x, \forall x \in S\} \quad \forall U \subset S$$

In summary, if I is admissible for φ, Let the base be the I-adic completion $\widehat{R}_I$. Then we have a family $(\mathcal{X}, \mathcal{L}, G, \varrho)$ over $\operatorname{Spec} \widehat{R}_I$ called the AN family. The coherent sheaf $\pi_*\mathcal{L}$ is locally free of rank d, and there is a Fourier decomposition $\pi_*\mathcal{L} \cong \oplus_{\alpha \in X/\phi(Y)} \pi_*\mathcal{M}_\alpha$. If we choose a section $\vartheta \in \oplus_{\alpha \in \mathscr{P}^0} \pi_*\mathcal{M}^*_\alpha$, then $(\mathcal{X}, \mathcal{L}, G, \varrho, \Theta)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}(\widehat{R}_I)$.

Analogously to Corollary 2.17, the AN construction is functorial. Assume S, S' are as above, with degeneration data, charts $P \to R$ and $P' \to R'$, and compatible functions $\varphi : X_{\mathbf{R}} \to P^{\mathrm{gp}}_{\mathbf{R}}$, $\varphi' : X_{\mathbf{R}} \to (P'_{\mathbf{R}})^{\mathrm{gp}}$. Suppose $(f, f^\flat)$ is a log morphism $S' \to S$ compatible with the charts $P \to R$ and $P' \to R'$, that is

$$\begin{tikzcd} P \arrow[r, "f^\flat"] \arrow[d] & P' \arrow[d] \\ R \arrow[r, "f^\sharp"] & R'. \end{tikzcd} \tag{4.5}$$

Moreover, the degeneration data over S' is isomorphic to the pull back of degeneration data along f, and $\psi := f^\sharp \circ \varphi - \varphi'$ takes values in $(R')^*$.

Proposition 4.19. *The pull back of the AN family $(\mathcal{X}, \mathcal{L}, G, \varrho)$ over S is isomorphic to the AN family $(\mathcal{X}', \mathcal{L}', G', \varrho')$ over S'.*

Proof. By the commutative diagram 4.5, we get a morphism $k[S(X) \rtimes_\varphi P] \otimes_{k[P]} R' \to k[S(X) \rtimes_{\varphi'} P'] \otimes_{k[P']} R'$, which induces the morphism $(\widetilde{P}^r)' \to \widetilde{P}^r \times_S S'$. By Corollary 2.17, this is an isomorphism. Moreover, the pull back of $\widetilde{G}$ is $\widetilde{G}'$ by assumption. Since the contracted product commutes with the pull back, we have the isomorphism $\widetilde{\mathcal{X}}' \to \widetilde{\mathcal{X}} \times_S S'$. By assumption, the degeneration data also commutes with the pull back, the isomorphism commutes with Y-action. $\square$

Given S as above, with degeneration data, a function $\varphi : X_{\mathbf{R}} \to P^{\mathrm{gp}}_{\mathbf{R}}$, and a chart $\alpha : P \to R$ such that $\alpha \circ \varphi$ is compatible with the degeneration data. We get the family $(\mathcal{X}_I, \mathcal{L}_I, G_I, \varrho_I)$ over R/I. Suppose there is a different chart $\alpha' : P \to R$, then $\psi := \alpha \circ \varphi / \alpha' \circ \varphi$ takes values in R^*. Therefore $\alpha' \circ \varphi$ is also compatible with the degeneration data. Since the graded sheaf of algebra $\mathcal{R}$ as a subsheaf of $\mathcal{S}$ is defined independent of the choice of the chart $\alpha : P \to R$, the set of admissible ideals are the same for the two charts. We get the same family $(\mathcal{X}_I, \mathcal{L}_I, G_I, \varrho_I)$ over R/I by using α' in the AN construction. Moreover the Fourier decomposition (4.4) are the same for α and α'.

Lemma 4.20. *The AN construction $(\mathcal{X}, \mathcal{L}, G, \varrho)$, and the Fourier decomposition $\pi^*\mathcal{L} = \oplus_{\alpha \in X/\phi(Y)} \pi_* \mathcal{M}_\alpha$ only depend on the log structure induced by $\alpha : P \to R$.*

Since a log structure is an étale sheaf, we can glue the above local models and generalize the base S.

Assumption 4.21. *Assume the log scheme (S, M_S) is log smooth over a base $(\overline{S}, \mathcal{O}^*_{\overline{S}})$. Moreover, assume that the base $\overline{S}$ is Noetherian, excellent, and integral.*

If $(\overline{S}, \mathcal{O}^*_{\overline{S}})$ is log regular, then (S, M_S) is log regular. By ([Ols08] Theorem 2.3.16), étale locally S is isomorphic to the product $\overline{S}' \times X_P$, with $\overline{S}'$ log regular. Denote the open subset where M_S is trivial by S°, and the boundary $S \backslash S^\circ$ by ∂S. By ([Kat94] Theorem 11.6), the log structure M_S is isomorphic to the divisorial log structure $M_{(S,\partial S)}$ on S. Therefore, the log scheme (S, M_S) is also denoted by $S^\dagger$.

We can define the degeneration data over S. It is defined to be an étale sheaf of data $(A, A^t, \widetilde{G}, \widetilde{G}^t, T, T^t, c, c^t, X, Y, \phi, \psi, \tau, \mathcal{M})$, such that for each étale affine neighborhood U of S, and the completion of U with respect to $U \times_S \partial S$, it is the degeneration data defined before. For any point $x \in S$, we can choose an affine étale neighborhood $U \to S$ such that $U = \operatorname{Spec} R$ is isomorphic to a product $X_P \times \overline{S}$ with some base scheme $\overline{S}$. A choice of such isomorphism gives a projection $U \to X_P$ and a chart $\alpha : P \to R$. Fix a piecewise affine function $\varphi : X_{\mathbf{R}} \to P^{\mathrm{gp}}_{\mathbf{R}}$ as above. By choosing the local chart, we can talk about whether φ is compatible with the degeneration data, and the set of admissible ideals.

Lemma 4.22. *Whether φ is compatible with the degeneration data over U, and the set of admissible ideals are both independent of the choice of the trivialization of U.*

Proof. This is because different trivializations give the same log structure. $\square$

Definition 4.23. The function φ and the log structure on S is said to be *compatible* with the degeneration data over S if they are compatible under any local trivialization.

Definition 4.24. A closed subscheme Z with support inside ∂S is called *admissible* if restricted to any affine étale open set U, under any local trivialization of U, Z corresponds to an admissible ideal.

Suppose there is degeneration data over the base S. With respect to the integral structure $(c^t \times c)^* \mathcal{P}_A^{-1}|_{(\lambda,\alpha)}$, $\tau(\lambda, \alpha)$ defines a locally principal fractional ideal of R, denoted by $I_{\lambda,\alpha}$. With respect to the integral structure $(c^t)^* \mathcal{M}^{-1}|_\lambda$, the trivialization $\psi(\lambda)$ defines a locally principal fractional ideal of R, denoted by I_λ. Since τ and ψ are compatible,

$$I_{\lambda+\mu} = I_\lambda \cdot I_\mu \cdot I_{\mu,\phi(\lambda)}.$$

Therefore, τ can be regarded as a bimultiplicative form $B : Y \times X \to \operatorname{Div} S$. The support of B, that is the subspace of S where $B \not\equiv 0$, is the degenerate locus. Since B is bimultiplicative, by choosing a basis of X and Y, we can see that the degenerate locus is a finite union of Cartier divisors $I_{\lambda,\alpha}$, and thus is a Cartier divisor of S. If the map φ, the log structure, and the trivialization τ are compatible, the Cartier divisor $I_{\lambda,\alpha}$ is the Cartier divisor defined by $b_t(\lambda, \alpha)$. In this case, the degenerate locus is ∂S. We can get the log structure just from the degeneration data.

Theorem 4.25. *Assume S satisfies Assumption 4.21. Assume there is degeneration data over S, and a piecewise affine function $\varphi : X_{\mathbf{R}} \to P^{\mathrm{gp}}_{\mathbf{R}}$, compatible with the degeneration data through the log structure of $S^\dagger$. Then for any admissible closed subscheme Z, we construct a family $(\mathcal{X}_Z, \mathcal{L}_Z, G_Z, \varrho_Z)$ over Z. Furthermore, there is a subsheaf of $\pi_*\mathcal{L}$ defined by the Fourier decomposition* (4.4) *such that for any Θ_Z defined by a section of this subsheaf, the family $(\mathcal{X}_Z, \mathcal{L}_Z, G_Z, \varrho_Z, \Theta_Z)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}(Z)$. This construction is functorial with respect to morphisms of admissible closed subschemes $Z' \to Z$.*

4.2 Standard Data

In this section, we specify the degeneration data for the versal families, and compare them with the standard construction in [Ols08]. Fix an X-invariant integral paving $\mathscr{P}$ of $X_{\mathbf{R}}$. By definition, $\mathscr{P}$ is obtained as the set of affine domains of some X-quasiperiodic real-valued piecewise affine function ψ.

Associated with $\mathscr{P}$ is a rational polyhedral cone $C(\mathscr{P})$ in the second Voronoi fan $\Sigma(X)$. The support of $\Sigma(X)$ is $\mathcal{C}(X)^{\mathrm{rc}} \subset \Gamma^2 U$. Let $CPA(\mathscr{P}, \mathbf{R})$ be the space of convex piecewise affine functions whose associated paving is coarser than $\mathscr{P}$[1]. Let $CPA^X(\mathscr{P}, \mathbf{R})$ be the subspace of X-quasiperiodic functions. The map $\psi \to Q$, which maps ψ to the associated quadratic form Q, identifies $CPA^X(\mathscr{P}, \mathbf{R})/Aff$ with $C(\mathscr{P})$. Let $C(\mathscr{P}, \mathbf{Z})$ be the image of integral convex functions in $CPA(\mathscr{P}, \mathbf{R})/Aff$ and $C^X(\mathscr{P}, \mathbf{Z})$ be the image of integral, X-quasiperiodic, convex functions in $C(\mathscr{P})$.

Define $H_{\mathscr{P}}$ as in [Ols08]. For any $\sigma_i \in \mathscr{P}$ define $N_i = S(\sigma_i)$. Define $N_{\mathscr{P}} := \varinjlim N_i$ (in the category of integral monoids). The cone $S(X) = \varinjlim N_i$ in the category of sets. By universal property of $S(X)$ we have a natural map $\tilde{\varphi}' : S(X) \to N_{\mathscr{P}}$. Since $\mathrm{Hom}(\varinjlim N_i, \mathbf{Z}) \cong \varprojlim(\mathrm{Hom}(N_i, \mathbf{Z}))$, The group $N_{\mathscr{P}}^{\mathrm{gp}}$ is equal to $PA(\mathscr{P}, \mathbf{Z})^*$.

Let $\widetilde{H}_{\mathscr{P}}$ be the submonoid of $N_{\mathscr{P}}^{\mathrm{gp}}$ generated by

$$\alpha * \beta := \tilde{\varphi}(\alpha) + \tilde{\varphi}(\beta) - \tilde{\varphi}(\alpha + \beta), \quad \forall \alpha, \beta \in C(X).$$

Recall in the proof of ([Ols08] Lemma 4.1.6), it is shown that the image of $\pi : SC_1(\mathbb{X}_{\geqslant 0})'/B_1 \to \widetilde{H}_{\mathscr{P}}$ is equal to $\widetilde{H}_{\mathscr{P}}$. By the same argument of Proposition 2.20, we have

Proposition 4.26. *Let $C(\mathscr{P}, \mathbf{Z})^\vee$ be the dual of the monoid $C(\mathscr{P}, \mathbf{Z})$ in the*

[1] Notice we define $CPA(\mathscr{P}, \mathbf{R})$ to be closed.

dual group $\mathrm{Hom}(C(\mathscr{P},\mathbf{Z}),\mathbf{Z})$.

$$\widetilde{H}^{\mathrm{sat}}_{\mathscr{P}} = C(\mathscr{P},\mathbf{Z})^{\vee}.$$

The group X is acting on $S(X)\subset \mathbb{X}$ by

$$(n,\alpha)\mapsto (n,\alpha+\beta) = (n,\alpha)+(0,n\beta) \quad \forall \beta\in X.$$

It induces an action of X on $N_{\mathscr{P}}$ and $\widetilde{H}_{\mathscr{P}}$. The quotient of this action is denoted by $H_{\mathscr{P}}$. There is a natural map

$$H_{\mathscr{P}} \to C^X(\mathscr{P},\mathbf{Z})^{\vee}.$$

Corollary 4.27. We have

$$\mathrm{Hom}(H_{\mathscr{P}},\mathbf{Z}_{\geqslant 0}) = C^X(\mathscr{P},\mathbf{Z}),$$

and

$$H^{\mathrm{sat}}_{\mathscr{P}}/(H^{\mathrm{sat}}_{\mathscr{P}})_{\mathrm{tor}} = C^X(\mathscr{P},\mathbf{Z})^{\vee}.$$

Proof. We prove $C^X(\mathscr{P},\mathbf{Z})\subset \mathrm{Hom}(H_{\mathscr{P}},\mathbf{Z}_{\geqslant 0})$ first. The inclusion $C^X(\mathscr{P},\mathbf{Z})\to C(\mathscr{P})\subset \Gamma^2 U$ induces a map $S^2X\to H^{\mathrm{gp}}_{\mathscr{P}}$, and this is the map $s: S^2X\to H^{\mathrm{gp}}_{\mathscr{P}}$ defined in ([Ols08] Lemma 5.8.2). By ([Ols08] Lemma 5.8.16), $C(\mathscr{P})^{\mathrm{gp}}_{\mathbf{Q}}\cong \mathrm{Hom}(H_{\mathscr{P}},\mathbf{Q})$. Therefore $C^X(\mathscr{P},\mathbf{Z})\subset \mathrm{Hom}(H_{\mathscr{P}},\mathbf{Z}_{\geqslant 0})$ is an inclusion. On the other hand, for each $\psi\in \mathrm{Hom}(H_{\mathscr{P}},\mathbf{Z}_{\geqslant 0})$, denote its image in $\mathrm{Hom}(\widetilde{H}_{\mathscr{P}},\mathbf{Z}_{\geqslant 0})$ by $\tilde{\psi}$. Since $\widetilde{H}^{\mathrm{sat}}_{\mathscr{P}} = C(\mathscr{P},\mathbf{Z})^{\vee}$, $\tilde{\psi}$ is an element in $C(\mathscr{P},\mathbf{Z})$. Being invariant under the action of X means exactly being X-quasiperiodic, therefore $\psi\in C^X(\mathscr{P},\mathbf{Z})$.

Both $C^X(\mathscr{P}, \mathbf{Z})^\vee$ and $H^{\mathrm{sat}}_{\mathscr{P}}/(H^{\mathrm{sat}}_{\mathscr{P}})_{\mathrm{tor}}$ are toric, and

$$\mathrm{Hom}(H_{\mathscr{P}}, \mathbf{Z}_{\geqslant 0}) = \mathrm{Hom}(H^{\mathrm{sat}}_{\mathscr{P}}/(H^{\mathrm{sat}}_{\mathscr{P}})_{\mathrm{tor}}, \mathbf{Z}_{\geqslant 0}).$$

So the second statement follows from the first statement. $\square$

Denote the monoid $C^X(\mathscr{P}, \mathbf{Z})^\vee$ by $P_{\mathscr{P}}$. It is a sharp toric monoid. The natural morphism $H^{\mathrm{sat}}_{\mathscr{P}} \to P_{\mathscr{P}}$ is the quotient out the torsion part, since $s_{\mathbf{Q}} : S^2X \otimes \mathbf{Q} \to H^{\mathrm{gp}}_{\mathscr{P},\mathbf{Q}}$ is an isomorphism ([Ols08] Proposition 5.8.15).

If the paving $\mathscr{P}$ is a triangulation $\mathscr{T}$, the cone $C(\mathscr{P})$ is of maximal dimension in $\Gamma^2 U$. The group $\mathrm{Hom}(C^X(\mathscr{P}, \mathbf{Z})^{\mathrm{gp}}, \mathbf{Z})$ is a lattice in S^2U^*, and is denoted by $\mathbb{L}_{\mathscr{P}}$. We have $S^2X \subset \mathbb{L}_{\mathscr{P}}$. For a general paving $\mathscr{P}$, let $I_{\mathscr{P}}$ denote the set of triangulations that refine $\mathscr{P}$. Define

$$\mathbb{L}_{\mathscr{P}} := \sum_{\mathscr{T} \in I_{\mathscr{P}}} \mathbb{L}_{\mathscr{T}}.$$

Let $C(\mathscr{P})^\vee$ be the dual cone of $C(\mathscr{P})$ in S^2U^*, and $S_{\mathscr{P}}$ be the toric monoid $C(\mathscr{P})^\vee \cap \mathbb{L}_{\mathscr{P}}$. If $\mathscr{P}$ is a triangulation, $S_{\mathscr{P}} = P_{\mathscr{P}}$.

Introduce the monoid $S(X) \rtimes H_{\mathscr{P}}$ on the set $S(X) \times H_{\mathscr{P}}$ with the addition law

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + \alpha * \beta).$$

The morphism $H^{\mathrm{sat}}_{\mathscr{P}} \to P_{\mathscr{P}}$ induces the natural morphism $S(X) \rtimes H^{\mathrm{sat}}_{\mathscr{P}} \to S(X) \rtimes P_{\mathscr{P}}$.

Recall the data from the mirror family. Fix the lattice X and the X-periodic paving $\mathscr{P}$ of $\overline{X}_{\mathbf{R}}$. Consider the minimal models of the mirror family

$\mathcal{Y}^0$ in the principally polarized case. By Theorem 3.50, we can identify the second Voronoi fan $\Sigma(X)$ with the Mori fan of $\mathcal{Y}^0$. For any bounded tilting $\mathscr{P}$, the nef cone $\overline{\mathcal{K}}(\mathcal{Y}^0_{\mathscr{P}}) \cap \mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}})$ is identified with $C^X(\mathscr{P}, \mathbf{Z})$. Therefore

$$P_{\mathscr{P}} = \overline{\mathrm{NE}}(\mathcal{Y}^0_{\mathscr{P}}) \cap \mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}})^*.$$

Define the piecewise affine multivalued, $\mathrm{NE}(\mathcal{Y}^0_{\mathscr{P}})$-convex function $\varphi_{\mathscr{P}} : B \to \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}}) = P^{\mathrm{gp}}_{\mathscr{P},\mathbf{R}}$ by requiring that, at each codimension-1 wall $\rho \subset C(\underline{B})$ the bending parameter $p_\rho \in \mathrm{NE}(\mathcal{Y}^0_{\mathscr{P}})$ is the corresponding curve class $\pi_*[V(C(\rho))]$, where $\pi : \widetilde{\mathcal{Y}}^0_{\mathscr{P}} \to \mathcal{Y}^0_{\mathscr{P}}$ is the universal covering.

Lemma 4.28. *Let ψ be an integral, X-quasiperiodic piecewise affine function whose paving is coarser than $\mathscr{P}$. Let ρ be a lift of the cone to the fan of the universal covering $C(\overline{X}_{\mathbf{R}})$. Define ψ_i, ψ_j and ω as before. The bending parameter p_ρ for $\varphi_{\mathscr{P}}$ is*

$$p_\rho(\psi) = \psi_i(\omega) - \psi_j(\omega).$$

In particular, the function $\varphi_{\mathscr{P}}$ is integral with respect to the integral structure $P_{\mathscr{P}}$.

Proof. Since ψ is X quasi-periodic, the divisor D_ψ it represents is inside $\mathrm{Pic}^X(\widetilde{\mathcal{Y}}^0_{\mathscr{P}})$. Therefore

$$p_\rho(\psi) = (\pi_*[V(C(\rho))]) \cdot D_\psi = \pi_*([V(C(\rho))] \cdot \pi^* D_\psi) = \psi_i(\omega) - \psi_j(\omega).$$

The last equality is by the formula for toric varieties, since $\widetilde{\mathcal{Y}}^0_{\mathscr{P}}$ is locally a toric variety.

If ψ is integral, $p_\rho(\psi)$ is an integer, since $\omega \in \overline{X}$. In other words, $p_\rho \in P_{\mathscr{P}}$, and $\varphi_{\mathscr{P}}$ is integral. $\square$

Corollary 4.29. For any function $\psi \in C(\mathscr{P})$, the evaluation $\varphi_{\mathscr{P}}$ on ψ is equal to ψ up to a linear function.

Proof. Choose a piecewise affine function ψ as the representative. By the Lemma 4.28, the bending parameters of $\psi \circ \varphi_{\mathscr{P}}$ is equal to the bending parameters of ψ. $\square$

If $C(\mathscr{P}')$ is a face of a cone $C(\mathscr{P})$. Consider the dual $C(\mathscr{P}')^\vee$ in the space $\mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}})$. Since $\mathscr{P}$ refines $\mathscr{P}'$, there is a contraction f inducing $f_* : \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}}) \to \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}'})$. We have an exact sequence of monoids

$$0 \longrightarrow \mathcal{R} \longrightarrow C(\mathscr{P}')^\vee \xrightarrow{f_*} \overline{\mathrm{NE}}(\mathcal{Y}^0_{\mathscr{P}'}) \longrightarrow 0,$$

where $\mathcal{R}$ is the $\mathbf{R}$-vector space generated by the contracted curve classes in the contraction $f : \mathcal{Y}^0_{\mathscr{P}} \to \mathcal{Y}^0_{\mathscr{P}'}$.

Corollary 4.30. The standard sections φ are compatible.

$$\varphi_{\mathscr{P}'} = f_* \circ \varphi_{\mathscr{P}} : B \to \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}'}).$$

Proof. Regard $\mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}})$ as the dual space of $\mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}})$, and $\mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}'}) \to \mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}})$ as an inclusion. The map f_* is the restriction of functions to the subspace $C(\mathscr{P}')$. By Lemma 4.28, the bending parameters have the same description as functionals on $C(\mathscr{P}')$. $\square$

Remark 4.31. The function f_* is just the quotient by the face $\mathcal{R}$.

Assume R is a Noetherian, excellent, normal integral domain complete with respect to an ideal I. Assume there exists an object in $\mathrm{DD}_{\mathrm{ample}}$ over $S = \operatorname{Spec} R$, and a chart $\alpha : P_{\mathscr{P}} \to R$ such that $\varphi_{\mathscr{P}}$ is compatible with degeneration data, and I is admissible for $\varphi_{\mathscr{P}}$. Do the AN construction and get $(\mathcal{X}, \mathcal{L}, G, \varrho)$.

Locally on A/S, choose a trivialization of $\mathcal{M}$ and compatible trivializations of $\mathcal{O}_\alpha$, the algebra $\mathcal{R}$ is isomorphic to $k[S(X) \rtimes P_{\mathscr{P}}] \otimes_{k[P_{\mathscr{P}}]} \mathcal{O}_A$. Define the log structure $\widetilde{P} \to \mathcal{O}_{\widetilde{\mathcal{X}}}$ locally by the descent of the chart

$$S(X) \rtimes P_{\mathscr{P}} \to k[S(X) \rtimes P_{\mathscr{P}}] \otimes_{k[P_{\mathscr{P}}]} \mathcal{O}_A.$$

For any $n \in \mathbf{N}$, do the reduction over $S_n := \operatorname{Spec} R/I^n$, the pull back $\widetilde{P}_n$ on $\widetilde{X}_n$ descends to a log structure P_n on the quotient X_n by ([Ols08] 4.1.18, Lemma 4.1.19, & 4.1.22).

By ([Ols08] Lemma 4.1.11), $(\widetilde{\mathcal{X}}, \widetilde{P})$ is integral and log smooth over $S^\dagger$. Since being log smooth and integral is a property stable under base change, $(\widetilde{X}_n, \widetilde{P}_n)$ are log smooth and integral over $S_n^\dagger$. Since $\widetilde{X}_n \to X_n$ is étale and surjective, and being log smooth is a property defined étale locally, (X_n, P_n) is log smooth over $S_n^\dagger$.

The underlying topological space of S_n inherits a stratification from the toric stratification of $\operatorname{Spec} k[P_{\mathscr{P}}]$. Since the image of $\operatorname{Spec} R/I$ is in the union of the toric strata defined by $I_{\varphi_{\mathscr{P}}}$, each strata corresponds to an admissible prime toric ideal J. Let J be an admissible prime toric ideal of $P_{\mathscr{P}}$, and F be the face $P_{\mathscr{P}} \setminus J$. The monoid $P_{\mathscr{P}}$ is the integral points in $C(\mathscr{P})^\vee$, for $C(\mathscr{P})$ a cone in the second Voronoi fan. Each face τ of $C(\mathscr{P})$ is a cone $C(\mathscr{P}')$ for some paving $\mathscr{P}'$ coarser than $\mathscr{P}$. Therefore $F = C(\mathscr{P}')^\perp \cap C(\mathscr{P})^\vee$ for some paving $\mathscr{P}'$. Then the stratum of S_n is called the stratum associated to the paving $\mathscr{P}'$, or the $\mathscr{P}'$-stratum.

We can associate a paving and a piecewise affine function with the face F as in Definition 4.4. By Corollary 4.30 and Remark 4.31, the piecewise affine function is $\varphi_{\mathscr{P}'}$, and the paving is $\mathscr{P}'$.

Lemma 4.32.

$$P_{\mathscr{P}}/(F \cap P_{\mathscr{P}}) = P_{\mathscr{P}'}.$$

Proof. It is the same with the argument in Lemma 2.51. The monoid $P_{\mathscr{P}}$ is given by the intersection $\overline{\mathrm{NE}}(\mathcal{Y}^0_{\mathscr{P}}) \cap \mathrm{Pic}(\mathcal{Y}^0_{\mathscr{P}})^*$. The dual integral structure is $C^X(\mathscr{P}, \mathbf{Z})^{\mathrm{gp}}$. We claim that $C^X(\mathscr{P}', \mathbf{Z}) = C(\mathscr{P}') \cap C^X(\mathscr{P}, \mathbf{Z})^{\mathrm{gp}}$. Assume ψ is a piecewise function and its image is in $C^X(\mathscr{P}', \mathbf{Z})$. Since $\mathscr{P}'$ is coarser than $\mathscr{P}$. Each top-dimensional cell $\sigma \in \mathscr{P}$ is contained in some top-dimensional cell $\sigma' \in \mathscr{P}'$ of $\mathscr{P}'$. Therefore, the restriction of ψ to σ is integral, and $\psi \in C(\mathscr{P}') \cap C^X(\mathscr{P}, \mathbf{Z})^{\mathrm{gp}}$. On the other hand, each top-dimensional cell $\sigma' \in \mathscr{P}'$ contains some top-dimensional cell $\sigma \in \mathscr{P}$. If ψ is integral on σ, it is integral on σ'. As a result, $\psi \in C^X(\mathscr{P}', \mathbf{Z})$. This proves the claim. It follows that the dual integral structures also agree. □

Lemma 4.33. *For each geometric point $\bar{x} \to S$ in interior of the $\mathscr{P}'$-stratum, the fiber $(\mathcal{X}_{n,\bar{x}}, \mathcal{L}_{n,\bar{x}}, P_{n,\bar{x}}, G_{n,\bar{x}}) \to (\bar{x}, M_{\bar{x}})$ is isomorphic to the collection of data obtained from the saturation of the standard construction defined in [Ols08].*

Proof. Denote $P_{\mathscr{P}}$ by P, and $P_{\mathscr{P}'}$ by P' in this proof. Assume the admissible prime toric ideal associated to $\mathscr{P}'$-stratum is J. Use the notations F, P_F, and J_F as in Lemma 4.5. By Lemma 4.32, $P_F \cong P' \oplus F^{\mathrm{gp}}$. Denote the geometric point $\bar{x} \to S \to \operatorname{Spec} k[P]$ by $\bar{y} \to \operatorname{Spec} k[P]$.

The fiber $(\mathcal{X}_{n,\bar{x}}, \mathcal{L}_{n,\bar{x}}, G_{n,\bar{x}})$ is isomorphic to the fiber $(\mathcal{X}_{\bar{x}}, \mathcal{L}_{\bar{x}}, G_{\bar{x}})$. Since the image of $\bar{x}$ is contained in the $\mathscr{P}'$-stratum, $\bar{y}$ is contained in the localization $\operatorname{Spec} k[P_F]$. Therefore, we can replace R by the base change $R' = k[P_F] \otimes_{k[P]} R$. As in the proof of Lemma 4.5, after the localization $k[P_F]$, the function $\varphi_{\mathscr{P}}$ is equal to $\varphi_{\mathscr{P}'} + \psi$, where ψ has bending parameters in F^{gp}. Up to a global affine function, we can assume that $\varphi_{\mathscr{P}} = \varphi_{\mathscr{P}'} \oplus \psi$ with the values in $P_F = P' \oplus F^{\mathrm{gp}}$. Since F^{gp} are sent to $(R')^*$, the pull back of the degeneration data is compatible with $\varphi_{\mathscr{P}'}$. Furthermore, the base change of $\widetilde{\mathcal{X}}$ can be constructed by using $P' \to P' \oplus F^{\mathrm{gp}} \to R'$ and $\varphi_{\mathscr{P}'}$. Therefore, the base change of $\mathcal{X}$ to R' is isomorphic to the AN construction by using $P' \to R'$ and $\varphi_{\mathscr{P}'}$.

Assume $\bar{x} = \operatorname{Spec} \Omega(x)$. Since the prime ideal of the image of $\bar{y}$ contains J_F which contains $P' \backslash \{0\}$, the prime ideal of the image of $\bar{x}$ contains $P' \backslash \{0\}$. Therefore, the map $H^{\mathrm{sat}}_{\mathscr{P}'} \to P' \to \Omega(x)$ sends nonzero elements to 0. Further pull back the degeneration data to $\Omega(x)$. Use the pull back of $b_t^{-1}\tau$ and $a_t^{-1}\psi$ as the trivializations, we get the data as in ([Ols08] 5.2.1). The fiber

$(\mathcal{X}_{\bar{x}}, \mathcal{L}_{\bar{x}}, G_{\bar{x}})$ is isomorphic to the standard construction in (Loc.cit. 5.2).

Over $S' = \operatorname{Spec} R'$, we can replace the chart P by the localization P_F. Since $P_F = P' \oplus F^{\mathrm{gp}}$ and F^{gp} are sent to $(R')^*$, we can use the chart $P' \to R'$. Pull back to $\Omega(k)$, $P' \to \Omega(x)$ is the chart. Define the map $\alpha : H^{\mathrm{sat}}_{\mathscr{P}'} \to P' \to \Omega(x)$. Since $(H^{\mathrm{sat}}_{\mathscr{P}'})_{\mathrm{tor}} \subset (H^{\mathrm{sat}}_{\mathscr{P}'})^*$ and is sent to $1 \in \Omega(x)^*$ by α, the push forward λ

$$\begin{tikzcd}
\alpha^{-1}\Omega(x)^* \arrow[r] \arrow[d] & H^{\mathrm{sat}}_{\mathscr{P}'} \arrow[d, "\lambda"'] \\
\Omega(x)^* \arrow[r] & (H^{\mathrm{sat}}_{\mathscr{P}'})^a
\end{tikzcd}$$

factors through $H^{\mathrm{sat}}_{\mathscr{P}'} \to P'$. Therefore $\alpha : H^{\mathrm{sat}}_{\mathscr{P}'} \to \Omega(x)$ is a chart of the pull back of the log structure M.

Similarly, over $\widetilde{\mathcal{X}}_{\bar{x}}$, the log structure $\widetilde{P}_{\bar{x}}$ is locally defined by the chart $S(X) \rtimes H^{\mathrm{sat}}_{\mathscr{P}'}$. Therefore, the log structures are also obtained from the saturation of the standard construction. $\square$

It follows that the family $(\mathcal{X}_n, \mathcal{L}_n, P_n, G_n, \varrho_n) \to (S_n, M_n)$ is an object in $\overline{\mathscr{T}}_{g,d}$. By the fact that $\overline{\mathscr{T}}_{g,d}$ is an Artin stack, we get a family $(\mathcal{X}, \mathcal{L}, P, G, \varrho) \to (S, M)$ in $\overline{\mathscr{T}}_{g,d}(S)$, and call this the standard family.

We notice that the log structure P on $\mathcal{X}$ does not depend on the choice of the chart $\alpha : P_{\mathscr{P}} \to R$. It follows from the proof of Lemma 4.33 that the open locus where P is trivial is the pull back of $\mathcal{X}$ along $S^\circ \to S$. Denote the complement by $\partial\mathcal{X}$. Since $(\mathcal{X}, P)$ is log smooth over $S^\dagger$, if $S^\dagger$ is log regular,

$(\mathcal{X}, P)$ is log regular by ([Kat94] Theorem 8.2). Then again by ([Kat94] Theorem 11.6), $(\mathcal{X}, P)$ is isomorphic to the divisorial log structure $\mathcal{M}_{\mathcal{X},\partial\mathcal{X}}$. Thus the log scheme $(\mathcal{X}, P)$ is also denoted by $\mathcal{X}^\dagger$.

Theorem 4.34. *The same assumption as in Theorem 4.25, with $P = P_{\mathscr{P}}$ and $\varphi = \varphi_{\mathscr{P}}$. If Z is a closed subscheme admissible for $\varphi_{\mathscr{P}}$, then there exists a family $(\mathcal{X}_Z, \mathcal{L}_Z, P_Z, G_Z, \varrho_Z)$ over $Z^\dagger$, which is an object in $\overline{\mathscr{T}}_{g,d}(Z)$. Here P_Z is induced from the divisorial log structure. Moreover, this construction is functorial with respect to $Z' \to Z$.*

The divisorial log structures are not preserved by the finite base change, unless the base change is étale. This is because the pull back of Cartier divisors f^* may map a prime divisor to a multiple of a prime divisor. However, the log structure P here is minimal in the sense that we take the minimal base change such that the central fiber is reduced.

4.3 Gluing Different Standard Families

Let $X^* = U \cap \Lambda$ and $X = \mathrm{Hom}(X^*, \mathbf{Z})$. Let $C(\mathscr{P}) \in \Sigma(X)$ be a polyhedral cone inside the interior of $\mathcal{C}(X) \subset \Gamma^2 U$. In other words, the paving $\mathscr{P}$ is bounded. Recall we have an integral structure $\mathbb{L}^*$ on $\Gamma^2(U)$ given by the integral tropical abelian varieties. Denote the dual lattice in S^2U^* by $\mathbb{L}$. This is the lattice for the coarse moduli space. If $\mathscr{T}$ is a minimal triangulation, $C(\mathscr{T})$ is a top-dimensional cone inside $\mathcal{C}(X)$. $\mathrm{N}_1(\mathcal{Y}^0_{\mathscr{T}})$ is identified with S^2U^*, and the cone $\overline{\mathrm{NE}}(\mathcal{Y}^0_{\mathscr{T}})$ is identified with $C(\mathscr{T})^\vee$. Notice $\mathbb{L} \subset S^2X \subset \mathbb{L}_{\mathscr{T}}$.

Corollary 4.35. Regard $\varphi_{\mathscr{P}}$ as a piecewise affine function $X_{\mathbf{R}} \to \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{P}})$. Then, up to an affine morphism, $\varphi_{\mathscr{P}}$ is the $\mathscr{P}$-affine interpolation of a quadratic map that factors through the universal quadratic function $\widetilde{Q} : X_{\mathbf{R}} \to S^2U^*$.

Proof. By Corollary 4.30, we only need to consider $\mathscr{P} = \mathscr{T}$ minimal triangulation. The universal quadratic function $\widetilde{Q} : X_{\mathbf{R}} \to S^2U^*$ is defined as

$$\widetilde{Q}(\alpha) = \frac{1}{2}\alpha \otimes \alpha.$$

For any quadratic function Q_ψ whose associated paving is $\mathscr{P}$, we have

$$\widetilde{Q}(\alpha)(Q_\psi) = 1/2(\alpha \otimes \alpha)(Q_\psi) = Q_\psi(\alpha) \equiv \psi(\alpha) \quad (\text{mod affine functions}), \quad \forall \alpha \in X.$$

By Corollary 4.29, $\varphi_{\mathscr{T}}(\alpha) = \widetilde{Q}(\alpha)$ for $\alpha \in X$, up to an affine morphism. Since $\mathscr{T}$ is a minimal triangulation, $\widetilde{Q}$ is the quadratic function associated to the X-quasi-periodic piecewise function $\varphi_{\mathscr{T}}$. $\square$

Choose a compatible basis $\{x^1, \ldots, x^r\}$ of the lattice X, and denote the simplex spanned by $\{0, x^1, \ldots, x^r\}$ by σ. The simplex σ is regular with respect to $\mathbb{X}$. Define the affine extension operator L_σ.

Definition 4.36. For any piecewise affine function ψ, $L_\sigma(\psi)$ is the affine extension to $X_{\mathbf{R}}$ of $\psi|_\sigma$.

Definition 4.37. The map $\Psi : X \to \mathbb{L}_{\mathbf{Q}}$ is defined by

$$\langle \overline{\psi}, \Psi(\omega) \rangle = \psi(\omega) - L_\sigma(\psi)(\omega), \quad \forall \overline{\psi} \in \mathbb{L}^*_{\mathbf{Q}}$$

Lemma 4.38. *The values of Ψ are in the lattice S^2X.*

Proof. Assume ψ is a piecewise affine function such that $\psi(\alpha) = 1/2B_\psi(\alpha,\alpha)+1/2L(\alpha)$ for $\alpha \in X$, with the associated symmetric bilinear form B_ψ in $\Gamma^2 X$ and L affine. For an arbitrary $\omega \in X$, let $\omega = \sum_i a_i x^i$ for $a_i \in \mathbf{Z}$. Compute

$$\begin{aligned}
\langle\overline{\psi}, \Psi(\omega)\rangle &= \psi(\omega) - L_\sigma(\psi)(\omega) \\
&= \frac{1}{2}B_\psi(\sum_i a_i x^i, \sum_j a_j x^j) + \frac{1}{2}L(\omega) - \sum_i a_i L_\sigma(\psi)(x^i) \\
&= \sum_{i\neq j} \frac{1}{2}a_i a_j (B_\psi(x^i,x^j) + B_\psi(x^j,x^i)) + \frac{1}{2}(\sum_i a_i^2 B_\psi(x^i,x^i) - \sum_i a_i B_\psi(x^i,x^i)) \\
&= \sum_{i\neq j} a_i a_j B_\psi(x^i \otimes x^j) + \sum_i \frac{1}{2}a_i(a_i-1)B_\psi(x^i\otimes x^i),
\end{aligned}$$

which is always an integer. $\square$

Corollary 4.39. For any $\lambda \in Y$, we have $2\Psi(\phi(\lambda)) \in \mathbb{L}$.

Proof. Since $2\Psi(\phi(\lambda_1)+\phi(\lambda_2)) = 2\Psi(\phi(\lambda_1)) + 2\Psi(\phi(\lambda_2)) + 2\phi(\lambda_1)\otimes\phi(\lambda_2)$, and $\phi(\lambda_1)\otimes\phi(\lambda_2) \in \mathbb{L}$ by the definition of $\mathbb{L}$, we only need to check it for a basis of Y. For example, we can check it for $\phi(\lambda_i) = d_i x^i$ since $\{x^1,\ldots,x^r\}$ is a compatible basis. By the computation above, for $B_\psi \in \mathbb{L}$, we have

$$\begin{aligned}
\langle B_\psi, 2\Psi(d_i x^i)\rangle &= d_i(d_i-1)B_\psi(x^i\otimes x^i) \\
&= (d_i-1)B_\psi(d_i x^i \otimes x^i) \\
&= (d_i-1)B_\psi(\phi(\lambda_i)\otimes x^i) \in \mathbf{Z}.
\end{aligned}$$

$\square$

If $\mathscr{T}$ is a minimal triangulation, $\mathrm{N}_1(\mathcal{Y}^0_{\mathscr{T}})$ is identified with S^2U^*. Since $\varphi_{\mathscr{T}}$ is only defined up to an affine map, and $S^2X \subset \mathbb{L}_{\mathscr{T}}$ for any $\mathscr{T}$, we can fix $\varphi_{\mathscr{T}}$ by requiring that $\varphi_{\mathscr{T}}|_X = \Psi$. We say $\varphi_{\mathscr{T}}|_X$ is canonical.

Lemma 4.40. *The functions $\varphi_{\mathscr{T}}$ thus defined are integral with respect to $\mathbb{L}_{\mathscr{T}}$.*

Proof. Choose any $\psi \in \mathrm{Pic}(\mathcal{Y}^0_{\mathscr{T}})$. We have

$$\langle \psi, \varphi_{\mathscr{T}} \rangle = \psi - L_\sigma(\psi).$$

The function ψ is integral with respect to $\mathscr{T}$ by definition. In particular $\psi(x^i)$ and $\psi(0)$ are all integers. Since $\{0, x^1, \ldots, x^r\}$ is a basis for $\mathbb{X}$, $L_\sigma(\psi)$ is an integral affine function. $\square$

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two minimal triangulations corresponding to maximal polyhedral cones $\sigma_1, \sigma_2 \in \Sigma(F)$. Let $\tau := C(\mathscr{P}) = \overline{\sigma}_1 \cap \overline{\sigma}_2$ be the cone of intersection. By the same argument as in Corollary 2.50, we only need to consider the case that σ_1 and σ_2 are adjacent.

Consider $\mathscr{P}$ as a X-periodic polyhedra paving of $X_{\mathbf{R}}$ given by $\psi \in \tau^\circ$. Notice that $X \subset \mathscr{P}$. By definition of the second Voronoi fan, $\mathscr{T}_i$ is a refinement of $\mathscr{P}$. Therefore the fan $\Sigma_{\mathscr{P}}$ is included in $\Sigma_{\mathscr{T}_i}$ as a subfan, and we have a refinement map from $\Sigma_{\mathscr{T}_i} \to \Sigma_{\mathscr{P}}$. This inclusion induces a morphism $\tilde{f}'_i : \widetilde{\mathcal{Y}}^0_{\mathscr{T}_i} \to \widetilde{\mathcal{Y}}^0_{\mathscr{P}}$ between toric embeddings. Since it is X-equivariant, it induces a birational morphism $f'_i : \mathcal{Y}^0_{\mathscr{T}_i} \to \mathcal{Y}^0_{\mathscr{P}}$.

Choose a convex fundamental domain Q' for the X-action such that Q' is a union of cells in $\mathscr{P}$. Take the closure of the fundamental domain, and denote it by Q. Restrict $\mathscr{T}_i$ also to Q, we are in the situation of the flipping case of Chap. 2 Sect. 2.3. Since $X \subset \mathscr{P}$, locally near Q, $\tilde{f}_i'$ is a toric flipping. Define everything as in the flipping case. Let $\mathscr{T}_1$ correspond to Σ_-. Define the wall $\varsigma := \sigma_{J\setminus\{k,l\}}$. A piecewise affine function ψ' corresponds to the pull back of an ample divisor π^*D if and only if ψ' is an affine interpolation of $A|_X$ for a quadratic function $A \in \overline{\sigma}_1 = \overline{\mathcal{K}}(\mathcal{Y}_{\mathscr{T}_1})$. By ([CLS11] Proposition 6.4.4), for such a ψ',

$$V(\varsigma)(\psi') = \frac{\mathrm{mult}(\varsigma)}{\mathrm{mult}(\sigma_k)(-b_k)}\Big(\sum_{i\in J_-}(-b_i)\psi'(v_i) - \sum_{i\in J_+} b_i\psi'(v_i)\Big).$$

Therefore $\pi_*[V(\varsigma)]\cdot D'_\psi = [V(\varsigma)](\psi') = 0$ exactly when $\psi' \in \tau = C(\mathscr{P})$, i.e. $\pi_*[V(\varsigma)] \in \mathcal{R} = \tau^\perp$. It follows that $f_i = f_i'$.

We call a birational contraction induced from such a toric morphism a toric flop contraction. We have proved

Lemma 4.41. *All birational contractions in the principally polarized case are toric flop contractions.*

Denote the functions $\varphi_{\mathscr{T}_i} : X_{\mathbf{R}} \to S^2U^*$ by φ^i. Define

$$g^{12} := \varphi^1 - \varphi^2.$$

The function g^{12} is a piecewise affine function $X_{\mathbf{R}} \to \mathrm{N}_1(\mathcal{Y}^0)$ and vanishes on X, since φ^i are both affine interpolation of Ψ. The cone $\tau^\vee \subset \mathrm{N}_1(\mathcal{Y}^0)$ is the localization of both $C(\mathscr{T}_i)$.

Lemma 4.42. *The values of g^{12} is in $(\tau^\vee)^*$.*

Proof. Consider the case when $\sigma_{\mathscr{T}_i}$ are adjacent, and τ is of codimension 1 cone. Consider the polytope Q as above. Restrict φ^i, g^{12} to Q. As in the GKZ case, define p_ρ by

$$\begin{aligned} g^{12} &:= \varphi^1 - \varphi^2 \\ &= \begin{cases} 0 & \text{if } x \in |\Sigma_{\mathscr{P}} \backslash \mathrm{Star}(\sigma_{J_- \cup J_+})| \\ \psi_{ij}(x) p_\rho & \text{if } x \in \sigma_{ij} \end{cases}, \end{aligned}$$

Compute p_ρ,

$$\begin{aligned} p_\rho &= g^{12}(\omega) \\ &= \varphi^1(\omega) - \varphi^2(\omega) \\ &= \sum_{i \in J_-} (-b_i)\Psi(v_i) - \sum_{i \in J_+} b_i \Psi(v_i) \end{aligned}$$

For any positive definite quadratic form $Q \in \mathcal{C}(X)$, let ψ' be the affine interpolation of $Q|_X$.

$$p_\rho(\psi') := p_\rho(Q) = \sum_{i \in J_-} (-b_i) Q(v_i) - \sum_{i \in J_+} b_i Q(v_i) = \sum_{i \in J_-} (-b_i)\psi'(v_i) - \sum_{i \in J_+} b_i \psi'(v_i). \tag{4.6}$$

Therefore

$$p_\rho = \frac{\mathrm{mult}(\sigma_k)(-b_k)}{\mathrm{mult}(\varsigma)} \pi_*[V(\varsigma)] \in \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{T}_i}).$$

Therefore p_ρ is a curve class in $\mathcal{R}$, and g^{12} takes values in $(\tau^\vee)^*$. □

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two minimal triangulations such that $C(\mathscr{T}_i)$, $i = 1, 2$ are maximal cones. Let $\tau = C(\mathscr{T}_1) \cap C(\mathscr{T}_2)$. Define $\mathbb{L}_\tau$ to be the lattice $\mathbb{L}_{\mathscr{T}_1} + \mathbb{L}_{\mathscr{T}_2}$ in S^2U^*. Since $\mathbb{L}_{\mathscr{T}_1}$ and $\mathbb{L}_{\mathscr{T}_2}$ are commensurable, $\mathbb{L}_\tau$ is commensurable to both of them. Define $S_\tau = (C(\mathscr{T}_1)^\vee + C(\mathscr{T}_2)^\vee) \cap \mathbb{L}_\tau$. Let $U_\tau := \operatorname{Spec} k[S_\tau]$. The inclusions of monoids $P_{\mathscr{T}_i} \to \tau^\vee \cap \mathbb{L}_{\mathscr{T}_i} \to S_\tau$ define morphisms $p_{\tau,\mathscr{T}_i} : U_\tau \to U_{C(\mathscr{T}_i)}$.

Lemma 4.43. *The morphisms $p_{\tau,\mathscr{T}_i} : U_\tau \to U_{C(\mathscr{T}_i)}$ are both étale for $i = 1, 2$.*

Proof. Denote the lattice $\mathbb{L}_{\mathscr{T}_i}$ by $\mathbb{L}_i$. Assume τ corresponds to the paving $\mathscr{P}$. By Lemma 4.32, we have the exact sequence,

$$0 \longrightarrow F \cap \mathbb{L}_i \longrightarrow \mathbb{L}_i \longrightarrow P_{\mathscr{P}} \longrightarrow 0.$$

Then copy the proof of Lemma 2.51, $p_{\tau,\mathscr{T}_i}$ is étale. $\square$

For any maximal cone $\sigma_\alpha \in \Sigma(F)$, we have the affine toric variety U_α. Denote $U := \coprod U_\alpha$. For each $\alpha \neq \beta$, assume $U_\alpha = U_{C(\mathscr{T}_i)}$ and $U_\beta = U_{C(\mathscr{T}_2)}$. Regard U_τ as the intersection $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2} : U_\tau \to U_\alpha \times U_\beta$. Denote $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2}$ by p_τ. For each α such that $\overline{\sigma}_\alpha = C(\mathscr{T})$, let $G_{\mathscr{T}}$ denote the kernel of the morphism $T_{\mathbb{L}^*_{\mathscr{T}}} \to T_\xi$. When the order of $G_{\mathscr{T}}$ is invertible in k, it is isomorphic to the constant group scheme $\mathbb{L}^*/\mathbb{L}^*_{\mathscr{T}}$, and is étale over k. Let R denote the étale equivalence relation generated by $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2}$ and $G_{\mathscr{T}}$ as in the GKZ case. Denote the algebraic stack $[U/R]$ by $X_{\Sigma(F)}$. For each $q \in B(\mathbf{Q})$, $g^{\alpha\beta}(q) \in \mathbb{L}_\tau \cap \tau^\perp$, and thus

$$\mathrm{X}^{g^{\alpha\beta}(q)} \in \mathrm{H}^0(U_\tau, \mathcal{O}^*).$$

If $\alpha = \beta$, define $\mathrm{X}^{g^{\alpha\beta}(q)}$ to be trivial. Since $g^{\alpha\beta}$ is defined by the difference, $\{\mathrm{X}^{g^{\alpha\beta}(q)}\}$ is a Čech 1-cocycle for the étale sheaf $\mathcal{O}^*$. It represents a line bundle, denoted by $\mathcal{L}_q$, over the algebraic stack $X_{\Sigma(F)}$.

4.4 Construction of the Stack

Fix the base ring $k = \mathbf{Z}[1/d, \zeta_M]/(F_M(\zeta_M))$, where $M = 2\delta_g$, F_M is the cyclotomic polynomial for the M-th roots of unity. We also write the primitive root of unit ζ_M as $\exp(2\pi i/M)$, and $\mathbf{Z}[1/d, \zeta_M]/(F_M(\zeta_M))$ as $\mathbf{Z}[1/d, \zeta_M]$.

Lemma 4.44. *The ring $k = \mathbf{Z}[1/d, \zeta_M]$ is a Dedekind domain of characteristic* 0.

Proof. The ring k is a integral domain with the field of fraction $K = \mathbf{Q}(\zeta_M)$. Since $\mathbf{Z}[\zeta_M]$ is the ring of algebraic integers in K, it is a Dedekind domain. As a localization of $\mathbf{Z}[\zeta_M]$, k is thus a Dedekind domain. □

In particular, the bases below would satisfy Assumption 4.21.

4.4.1 Infinitesimal Families over the 0-Cusps

Let $\Sigma(F)$ be the fan associated with the 0-cusp F. Associated with the 0-cusp F is a maximal isotropic subspace $U \subset V$ with respect to the symmetric bilinear form E. The lattice Y is $\Lambda/\Lambda \cap U^\perp$ and $X = \mathrm{Hom}(\Lambda \cap U, \mathbf{Z})$. The pairing E induces the homomorphism $\phi : Y \to X$ of type $\mathfrak{d}$. Since the degeneration is maximal, the abelian part A and the ample line bundle $\mathcal{M}$ are trivial. The trivializations τ and ψ can be written as functions $b : Y \times X \to K$ and $a : Y \to K$. Recall by choosing a basis of Λ, we can write

$$E = \begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix}$$

Recall the following data from Appendix 2.1.

$$b'(\lambda, \alpha) = \exp\left(-\pi i x' S \mathfrak{d}^{-1} \begin{pmatrix} v_1(\alpha) \\ \vdots \\ v_g(\alpha) \end{pmatrix} \right). \tag{4.7}$$

$$a'(\lambda) := \chi^{-1}(\lambda). \tag{4.8}$$

These data doesn't depend on the choice of the basis, but on a lift of Y to Λ. Furthremore, a' depends on the choice of a quadratic form χ.

Fix $K_0 := k[S^2 X]$. Define the tautological functions.

$$b_t : Y \times X \to K_0 \tag{4.9}$$

$$b_t(\lambda, \alpha) = \mathrm{X}^{\phi(\lambda) \otimes \alpha} \tag{4.10}$$

$$a_t : Y \to K_0^* \tag{4.11}$$

$$a_t(\lambda) = \mathrm{X}^{\Psi(\phi(\lambda))}. \tag{4.12}$$

Further define $b : Y \times X \to K_0$ to be $b = b_t b'$ and $a : Y \to K_0$ to be $a = a_t a'$.

We construct the families over all the 0-strata after the blowup. Each 0-stratum corresponds to a maximal cone $C(\mathscr{T})$ in $\mathcal{C}(X)$, associated to a minimal triangulation $\mathscr{T}$. Use $C(\mathscr{T})^\vee \subset S^2U^*$ and $P_{\mathscr{T}} = C(\mathscr{T})^\vee \cap \mathbb{L}_{\mathscr{T}}$. The base is $\operatorname{Spec}(P_{\mathscr{T}} \to k[P_{\mathscr{T}}])$. The function $\varphi_{\mathscr{T}} : U^* \to S^2U^* = \mathrm{N}_1(\mathcal{Y}^0_{\mathscr{T}})$ is integral with respect to $\mathbb{L}_{\mathscr{T}}$, and is compatible with the degeneration data a, b. Construct the standard family $(\mathcal{X}_I, \mathcal{L}_I, P_I, \varrho_I) \to (\operatorname{Spec} R_I, M_I)$ for every admissible ideal I. The polarization $(\mathcal{X}_I, \mathcal{L}_I)$ is of type δ. Since, over $\mathbf{Z}[1/d]$ the type of polarization is constant over a connected base, it suffices to check the type on a $\mathbf{C}$-point. By Appendix 2.1, we know the degeneration data gives the polarization of type δ. Since the construction is functorial for $I \subset I'$, we get an algebraic family over the formal base $\widehat{R}_I$ by Grothendieck's existence theorem ([GD63] EGA III_1 5.4.5).

Remark 4.45. As log schemes, the isomorphism class of $(\mathcal{X}_I, P_I, \varrho_I) \to (\operatorname{Spec} R_I, M_I)$ does not depend on the choice of a', and is thus canonical. The line bundle $\mathcal{L}_I$ depends on the choice. If the choice is changed, the $\mathcal{L}_I$ will be twisted by a line bundle of order 2. After a translation by an element in G_I, the lines bundles are isomorphic. Therefore, up to isomorphisms, the family $(\mathcal{X}_I, \mathcal{L}_I, P_I, \varrho_I, G_I) \to (\operatorname{Spec} R_I, M_I)$ is independent of the choice of the matrix S and the data a'.

Although it is not necessary for the construction of the stack, we show how the families are glued over the 0-cusp, when the order of $G_{\mathscr{T}}$ are all invertible in the base ring k. It suffices to make a finite base change of k because there are only finitely many different $C(\mathscr{T})$ up to the group action $\mathrm{GL}(X, Y)$.

Construct the stacky toric embedding $\overline{\Xi}(F) = X_{\Sigma(F)}$ from the fan $\Sigma(F)$, the étale morphisms $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2}$ and the étale groups $G_{\mathscr{T}}$. Define $\mathcal{O}_F$ to be the union of the boundary toric stratas $\mathfrak{Z}_\sigma$ for each $\sigma \in \Sigma(F)$ being a convex polyhedral cone in the interior of $\mathcal{C}(X)$. We can glue the families over the formal completion $\widehat{\Xi}(F)$ of $\overline{\Xi}(F)$ along the boundary $\mathcal{O}_F$. The étale base changes of $\overline{\Xi}(F)$ and $\widehat{\Xi}(F)$ are again denoted by $\overline{\Xi}(F)$ and $\widehat{\Xi}(F)$. We are free to make an étale base change when constructing a miniversal family.

Let Z be a closed subscheme of $X_{\Sigma(F)}$ whose image is contained in $\mathcal{O}_F$, and does not intersect the closed toric stratum corresponding to cones in the boundary of $\mathcal{C}(X)$. Let the ideal sheaf over the small étale site be $\mathscr{I}_Z$. To simplify the notations, for any étale map $U \to X_{\Sigma(F)}$, we denote $Z \times_{X_{\Sigma(F)}} U$ by $Z \cap U$.

Theorem 4.46. *For any such Z, there exists a projective, flat family π : $(\mathcal{X}_Z, \Theta_Z,$
$P_Z, \mathcal{O}(1), \varrho) \to (Z, M_Z)$, and a finite choices of canonical sections $\vartheta_q \in \pi_*(\mathcal{O}(1)) \otimes$ $\mathcal{L}_q$ for $q \in B(1/m)$ with the following properties*

1. *Restrict to each $\Xi_F(\mathscr{T}) \cap Z$, it is isomorphic to the family constructed above.*

2. *The family $\pi : (\mathcal{X}_Z, P_Z, \mathcal{L}_Z, \varrho) \to (Z, M_Z)$ is an object in $\overline{\mathscr{T}}_{g,d}$.*

3. *If $Z' \to Z$ is closed embedding, $(\mathcal{X}_{Z'}, P_{Z'}, \mathcal{L}_{Z'}, \varrho')$ is the pull back of $(\mathcal{X}_Z, P_Z,$
$\mathcal{L}_Z, \varrho)$.*

4. $\pi_*\mathcal{L}_Z^m$ *is locally free, and* $R^i\pi_*\mathcal{L}_Z^m = 0$*, for each* $m > 0, i > 0$*.*

5. Étale locally, for fixed m*, there exists a basis* $\{\vartheta_q\}_{q\in B(1/m)}$ *of global sections for* $\pi_*\mathcal{L}_Z^m \otimes \mathcal{L}_q$*. This basis is canonical up to constants in* μ_M*. In particular, étale locally, the collection* $\{\vartheta_q\}$ *for* $q \in B(\mathbf{Z})$ *is a basis of global sections for* $\pi_*\mathcal{L}_Z$*.*

Proof. For a maximal cone $\sigma_\alpha \in \mathcal{C}(X)$ corresponding to a minimal triangulation $\mathscr{T}$, let $U_\alpha = \Xi_F(\mathscr{T})$ be the associated affine open toric variety. The collection $\{U_\alpha\}$ is an affine étale open covering of $\overline{\Xi}(F)$. For any $\alpha \neq \beta$, denote $U_\alpha \times_{\overline{\Xi}(F)} U_\beta$ by U_τ, for $\tau = \overline{\sigma}_\alpha \cap \overline{\sigma}_\beta$. The closed subscheme Z is determined by the ideal sheaf $\mathscr{I}_Z$, and $\mathscr{I}_Z$ is specified by the collection of ideals $\{I_\alpha \subset k[P_{\mathscr{T}}]\}$ for $Z \cap U_\alpha \neq \emptyset$, such that the pull back to U_τ are equal as subsheaves of $\mathcal{O}_\tau$.

Fix a U_α such that $Z_\alpha := Z \cap U_\alpha \neq \emptyset$. We claim that $I_\alpha \in \mathfrak{I}_\varphi$, for $\varphi = \varphi_{\mathscr{T}}$. For simplicity, we use I instead of I_α in this paragraph. Since $\mathfrak{I}_\varphi$ is closed under finite intersection, we can assume that $\sqrt{I}$ is prime. Further, since the set theoretic image of Z is contained in the toric boundary $\mathcal{O}_F$, and doesn't intersect the closed toric stratum corresponding to cones in the boundary of $\mathcal{C}(X)$, $\sqrt{I}$ contains a prime toric ideal J. It suffices to prove that $J \in \mathfrak{I}_\varphi$. Since J is toric, it corresponds to a cone $C(\mathscr{P})$ in the interior of $\mathcal{C}(X)$, with $\mathscr{P}$ bounded, and coarser than $\mathscr{T}$. By Remark 4.31 The associated paving with this face is $\mathscr{P}$. Therefore J is admissible, and $I \in \mathfrak{I}_\varphi$.

Use the family $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, P_\alpha, \varrho_\alpha) \to (\operatorname{Spec} k[P_{\mathscr{T}}]/I_\alpha, M_\alpha)$ constructed above for I_α. Denote $k[P_{\mathscr{T}}]/I_\alpha$ by R_α, and the graded R_α-algebra $R_\alpha \otimes_{k[P_{\mathscr{T}}]}$

$k[S(X) \rtimes P_{\mathscr{T}}]$ by $\mathcal{R}_\alpha$. The total space $\widetilde{\mathcal{X}}_\alpha = \operatorname{Proj} \mathcal{R}_\alpha$.

Consider the group action of $G_{\mathscr{T}}$ on $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, P_\alpha, \varrho_\alpha) \to (\operatorname{Spec} k[P_{\mathscr{T}}]/I_\alpha, M_\alpha)$. Let ζ be an S-point of $G_{\mathscr{T}}$ for some k-scheme S. The element ζ is acting on the algebra $R_S := \mathcal{O}_S[P_{\mathscr{T}}]$ by

$$g : \mathrm{X}^p \mapsto \mathrm{X}^p(\zeta)\mathrm{X}^p.$$

Lemma 4.47. *There exists a bi-multiplicatively linear form $B_{\mathscr{T}} : \mathbb{X} \times G_{\mathscr{T}} \to k^*$ such that*

$$B_{\mathscr{T}}((1, \phi(\lambda)), \zeta) = \mathrm{X}^{\Psi(\phi(\lambda))}(\zeta) = a_t(\lambda)(\zeta).$$

Proof. First notice that since $\phi(\lambda) \otimes \omega \in \mathbb{L}$, we have

$$b_t(\lambda, \alpha)(\zeta) = \mathrm{X}^{\phi(\lambda)\otimes\omega}(\zeta) = 1$$

for any $\zeta \in G_{\mathscr{T}}$. Therefore $a_t(\lambda)(\zeta)$ is bi-multiplicatively linear on $Y \times G_{\mathscr{T}}$, and it is possible to extend a_t to $B_{\mathscr{T}}$.

It suffices to define $B_{\mathscr{T}}$ for the basis of $\mathbb{X}$, $\{(1,0), (1,x^1), \ldots, (1,x^g)\}$ in Definition 4.37. By Lemma 4.39,

$$\Psi(\phi(\lambda_i)) = \frac{1}{2} d_i(d_i - 1) x^i \otimes x^i \equiv \begin{cases} 0 \pmod{\mathbb{L}} & \text{if } d_i \text{ odd,} \\ 1/2 d_i x^i \otimes x^i \pmod{\mathbb{L}} & \text{if } d_i \text{ even.} \end{cases}$$

Define

$$B_{\mathscr{T}}((1, x^i), \zeta) = \begin{cases} 1 & \text{if } d_i \text{ odd,} \\ \mathrm{X}^{1/2 x^i \otimes x^i}(\zeta) & \text{if } d_i \text{ even.} \end{cases}$$

Since $(\mathrm{X}^{1/2x^i\otimes x^i}(\zeta))^{2d_i}=1$ and $2d_i|2\delta_g=M$, $\mathrm{X}^{1/2x^i\otimes x^i}(\zeta)$ is well-defined in k. Moreover, define $B_{\mathscr{T}}((1,0),\zeta)$ to be 1 for every $\zeta\in G_{\mathscr{T}}$. This completes the definition of $B_{\mathscr{T}}$. □

The pullback $(g^*\widetilde{\mathcal{X}}_\alpha, g^*\widetilde{\mathcal{L}}_\alpha)$ is $\operatorname{Proj} R_S/I_\alpha\otimes_{R_S}\mathcal{O}_S[S(Q_{\varphi_{\mathscr{T}}})]$. We regard $S(Q_{\varphi_{\mathscr{T}}})$ as a submonoid of $\mathbb{X}\times\mathbb{L}_{\mathscr{T}}$. Then $(g^*\widetilde{\mathcal{X}}_\alpha, g^*\widetilde{\mathcal{L}}_\alpha)$ is isomorphic to $(\widetilde{\mathcal{X}}_\alpha,\widetilde{\mathcal{L}}_\alpha)$ by an isomorphism over R_S/I_α

$$\begin{aligned} F_\zeta: g^*\mathcal{R}_{\mathscr{T}} &\longrightarrow \mathcal{R}_{\mathscr{T}},\\ 1\otimes\mathrm{X}^{(m,\omega,p)} &\longmapsto \mathrm{X}^p(\zeta)B_{\mathscr{T}}(-(m,\omega),\zeta)\mathrm{X}^{(m,\omega,p)}.\end{aligned}$$

To check whether F_ζ commute with the action S^*_λ of Y, i.e. , whether

$$F_\zeta\circ g^*(\psi(\lambda)^m\tau(\lambda,\omega))=\psi(\lambda)^m\tau(\lambda,\omega)\circ F_\zeta.$$

It is equivalent to check if

$$a_t^m(\lambda)b_t(\lambda,\omega)(\zeta)B_{\mathscr{T}}(-(m,\omega+m\phi(\lambda)),\zeta)=B_{\mathscr{T}}(-(m,\omega),\zeta). \tag{4.13}$$

However, $\phi(\lambda)\otimes\omega\in\mathbb{L}$ and thus $b_t(\lambda,\omega)(\zeta)=1$. Moreover $B_{\mathscr{T}}$ is bi-multiplicatively linear,

$$B_{\mathscr{T}}((m,m\phi(\lambda),\zeta)=B^m_{\mathscr{T}}((1,\phi(\lambda)),\zeta)=a_t^m(\lambda).$$

Therefore Equation (4.13) holds.

It follows that F_ζ induces an isomorphism between $(g^*\mathcal{X}_\alpha, g^*\mathcal{L}_\alpha)$ and $(\mathcal{X}_\alpha,\mathcal{L}_\alpha)$.

Remark 4.48. Notice that F_ζ is different from the definition in Proposition 2.47, and for generic fibers, it does not induce a homomorphism between abelian varieties. In general, if we want the isomorphism to preserve the line bundle, we have to interpret the proper scheme as the torsor over abelian schemes. If we use F_ζ in Proposition 2.47, then we get an isomorphism between polarized abelian varieties, but it does not preserve the line bundle.

Since the X-grading is preserved, the group action ϱ of $T = G_\alpha$ is preserved under F_ζ. The log structures are also preserved by F_ζ. It follows from the fact that the log structures are divisorial log structure, and F_ζ preserves the divisors.

For each $\omega \in X/\phi(Y)$, consider

$$\vartheta_\omega^\alpha := \sum_{\lambda \in Y} S_\lambda^*(\mathrm{X}^{\omega, \varphi_{\mathscr{T}}(\omega)}\theta).$$

Since I_α is admissible, this is a finite sum over R_α, and it defines a global section of $\mathcal{L}_\alpha$, also denoted by ϑ_ω^α. They are linearly independent, so $\{\vartheta_\omega^\alpha\}_{\omega \in X/\check{Y}}$ is a basis for $\mathrm{H}^0(\mathcal{X}_\alpha, \mathcal{L}_\alpha)$. For any $q \in B(1/m)$, we can similarly define ϑ_q^α, and $\{\vartheta_q^\alpha\}_{q \in B(1/m)}$ is a basis for $\mathrm{H}^0(\mathcal{X}_\alpha, \mathcal{L}_\alpha^m)$.

Under F_ζ, $g^*\vartheta_q^\alpha$ is mapped to

$$\mathrm{X}^{m\varphi_{\mathscr{T}}(q)}(\zeta)B_{\mathscr{T}}(-(m, mq), \zeta)\sum_{\lambda \in Y} S_\lambda^*(\mathrm{X}^{mq, m\varphi_{\mathscr{T}}(q)}\theta^m) = \mathrm{X}^{\sum a^i\Psi(\omega_i)}(\zeta)B_{\mathscr{T}}(-(m, mq), \zeta)\vartheta_q^\alpha$$

for $mq = \sum a^i\omega_i$. Since $\mathrm{X}^{\sum a^i\Psi(\omega_i)}(\zeta)B_{\mathscr{T}}(-(m, mq), \zeta)$ is a root of unity in μ_M, the section ϑ_q^α is well-defined up to a constant in μ_M.

Define the section $\vartheta^\alpha = \sum_{\omega \in X/\check{Y}} \vartheta^\alpha_\omega$, and Θ_α to be the zero locus of ϑ^α. The pair $(\mathcal{X}_\alpha, \Theta_\alpha)$ is stable. $\vartheta^\alpha_q, \vartheta^\alpha, \Theta_\alpha$ are not preserved by the action of $G_{\mathscr{T}}$. Therefore, they are only defined on the étale neighborhood Z_α.

Consider the intersection $U_\tau = U_\alpha \times_{\Xi(F)} U_\beta$. Do localization of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ respectively and regard them as algebras over $Z_\tau := U_\tau \times_{\Xi(F)} Z$. Use the model $\mathcal{R}_\alpha = R_\alpha \otimes_{k[P_{\mathscr{T}}]} k[S(X) \rtimes P_{\mathscr{T}}]$. Let $\mathrm{X}_\alpha^{(\omega,p)}$ be a monoid in $\mathcal{R}_\alpha$, and $\mathrm{X}_\beta^{(\omega,p)}$ be the corresponding monoid in $\mathcal{R}_\beta$. Twist $\mathcal{R}_\beta$ by the monoid of line bundles $\bigoplus \mathcal{L}_q$. That is, over $U_{\alpha\beta}$, define a map

$$\begin{aligned} \varphi_{\alpha\beta} : \mathcal{R}_\alpha &\longrightarrow \mathcal{R}_\beta \\ \mathrm{X}_\alpha^{(\omega,p)} &\longmapsto \mathrm{X}_\beta^{(\omega, p+\deg(\omega) g^{\alpha\beta}(\omega))}. \end{aligned}$$

As in the GKZ case, this is an isomorphism of algebras. Therefore, together with the isomorphisms $\{F_\zeta\}$, we can glue $\{\mathcal{R}_\alpha\}$ so that $\widetilde{\mathcal{R}} \otimes (\oplus \mathcal{L}_q)$ is a sheaf of graded algebras on Z. Construct $\widetilde{\mathcal{X}}_Z := \mathbf{Proj}\, \widetilde{\mathcal{R}} \otimes (\oplus \mathcal{L}_q)$.

The gluing $\varphi_{\alpha\beta}$ also commutes with the Y-actions. Assume $\deg(\omega) = m$ for $\omega \in S(X)$,

$$\begin{aligned} S_\lambda^*(\mathrm{X}_\alpha^{(\omega,p)}) &= a'(\lambda)^m b'(\lambda, m\omega) \mathrm{X}_\alpha^{(\omega+\phi(\lambda),p)} && (4.14) \\ &= a'(\lambda)^m b'(\lambda, m\omega) \mathrm{X}_\beta^{(\omega+\phi(\lambda), p+mg^{\alpha\beta}(\omega+\phi(\lambda)))} && (4.15) \\ &= a'(\lambda)^m b'(\lambda, m\omega) \mathrm{X}_\beta^{(\omega+\phi(\lambda),p)} \mathrm{X}^{mg^{\alpha\beta}(\omega+\phi(\lambda))} && (4.16) \\ &= S_\lambda^*(\mathrm{X}_\beta^{(\omega,p)} \mathrm{X}^{mg^{\alpha\beta}(\omega+\phi(\lambda))}) && (4.17) \\ &= S_\lambda^*(\mathrm{X}_\beta^{(\omega,p)} \mathrm{X}^{mg^{\alpha\beta}(\omega)}). && (4.18) \end{aligned}$$

In the last step, we claim that $g^{\alpha\beta}(\omega+\phi(\lambda)) = g^{\alpha\beta}(\omega)$. Since φ_α and φ_β are interpolation of the same quadratic function,

$$\varphi_\alpha(\omega+\phi(\lambda)) - \varphi_\alpha(\omega) = \varphi_\beta(\omega+\phi(\lambda)) - \varphi_\beta(\omega)$$

By definition $g^{\alpha\beta} = \varphi_\alpha - \varphi_\beta$, and we have $g^{\alpha\beta}(\omega+\phi(\lambda)) = g^{\alpha\beta}(\omega)$.

It follows that the quotients $\{(\mathcal{X}_\alpha, \mathcal{L}_\alpha)\}$ are glued together by $\varphi_{\alpha\beta}$ and $G_{\mathscr{T}}$, and we get $(\mathcal{X}_Z, \mathcal{L}_Z)$ over Z. Since the X-grading is preserved, the group action ϱ of $T = G_I$ is preserved by the gluing.

The log structures are also preserved under gluing. It follows from the fact that the log structures are divisorial log structure, and the gluing preserves the divisors. Notice that if the cone τ corresponds to the paving $\mathscr{P}$, then the points in the interior of Z_τ have charts $P_{\mathscr{P}}$ by the proof of Lemma 4.33. The fibers over these points have charts $S(X) \rtimes P_{\mathscr{P}}$. In sum, we get $\pi : (\mathcal{X}_Z, P_Z, \mathcal{L}_Z, \varrho) \to (Z, M_Z)$, and the conditions 1),2) are satisfied.

The AN construction of families over U_α is compatible with the base change $Z' \to Z$. Furthermore, the twists by $\oplus \mathcal{L}_q$ commute with the base change. Therefore 3) follows.

Under the isomorphism $\varphi_{\alpha\beta}$, the sections $\vartheta_q^\alpha = \vartheta_q^\beta \mathrm{X}^{\deg(q) g^{\alpha\beta}(q)}$ are identified after the twist of $\bigoplus \mathcal{L}_q$. We get global sections ϑ_q of the sheaf $\pi_* \mathcal{L}^m \otimes \mathcal{L}_q$ for $q \in B(1/m)$. If $q \in B(\mathbf{Z})$, there is no twist, ϑ_q is a section of $\pi_* \mathcal{L}$. However, they are not preserved by the group actions $\{G_{\mathscr{T}}\}$. This proves 5). Finally 4) follows from Corollary 4.18. $\square$

For every closed subscheme of $\widehat{\Xi}(F)$, we have assigned an object in $\overline{\mathscr{T}}_{g,d}$. By Theorem 4.46 3), this assignment is functorial. Moreover, we already know the pro-family is effective over complete local rings. Therefore, it defines a family in $\overline{\mathscr{T}}_{g,d}$ over the formal stack $\widehat{\Xi}(F)$. Let it be denoted by $(\mathcal{X}^f(F), P_F, \mathcal{L}^f, \varrho^f)$ over $\widehat{\Xi}(F)^\dagger$. Étale locally, we also have $(X^f(F), \mathcal{L}^f, \Theta^f, \varrho^f)$ in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.

Recall the discrete group $\overline{P}(F)$ is a subgroup of $\mathrm{GL}(X,Y)$ of finite index. $\mathrm{GL}(X,Y))$ is acting on the 1-dimensional family $\mathcal{Y} \to \Delta$ birationally, and fixing the base. The action is induced from the natural action of $\mathrm{GL}(X,Y)$ on the character group X. The birational action also induces the action of $\mathrm{GL}(X,Y)$ on $\mathrm{NS}(Y_t) = \Gamma^2 U$, which is the natural action on the quadratic forms via all the identifications. For any minimal model $\mathcal{Y}_{\mathscr{T}}$, regard the section φ as piecewise affine function $X_{\mathbf{R}} \to S^2U^*$, it is $\mathrm{GL}(X,Y)$-equivariant, and thus $\overline{P}(F)$-equivariant. That implies the construction is compatible with the action of $\overline{P}(F)$. Since $\mathrm{GL}(X,Y)$ preserves X and $\check{Y}$, the set $B(\mathbf{Z})$ and the coset of $\check{Y}$ is preserved.

Proposition 4.49. *The family $(\mathcal{X}^f(F)^\dagger, \mathcal{L}^f, \varrho^f) \to \widehat{\Xi}(F)^\dagger$ is $\overline{P}(F)$-equivariant.*

4.4.2 Infinitesimal Families over General Cusps

For the general case, denote the cusp by F_ξ, and we decorate every notation associated to this cusp by ξ. So the isotropic subspace is U_ξ of dimension r. Let $X_\xi^* = U_\xi \cap \Lambda$ and $X_\xi = \mathrm{Hom}(X_\xi^*, \mathbf{Z})$. Choose a basis $\{v_1, \ldots, v_r\}$ of X_ξ^*. Define $Y_\xi = \Lambda / U_\xi^\perp \cap \Lambda$. E defines the polarization ϕ :

$Y_\xi \to X_\xi$. Lift Y_ξ to Λ. Assume the basis is $\{u'_1, \ldots, u'_r\}$. Restrict the skew-symmetric pairing E to $\{u'_1, \ldots, u'_r\}$, we have a skew-symmetric integral matrix S_ξ. Choose a symmetric integral matrix S'_ξ such that $S'_\xi \equiv S_\xi \pmod{2\mathbf{Z}}$. Define the twist data

$$b' : Y_\xi \times X_\xi \to k, \quad a' : Y_\xi \to k, \tag{4.19}$$

$$b'(\lambda, \alpha) = \exp\left(-\pi i (x'_1(\lambda), \ldots, x'_r(\lambda)) S_\xi \mathfrak{d}^{-1} \begin{pmatrix} v_1(\alpha) \\ \vdots \\ v_r(\alpha) \end{pmatrix} \right), \tag{4.20}$$

$$a'(\lambda) = \exp\left(-1/2\pi i (x'_1(\lambda), \ldots, x'_r(\lambda)) S'_\xi \begin{pmatrix} x'_1(\lambda) \\ \vdots \\ x'_r(\lambda) \end{pmatrix} \right), \tag{4.21}$$

where $\{x'_1, \ldots, x'_r\}$ are the coordinates on Y_ξ with respect to the basis $\{u'_1, \ldots, u'_r\}$.

Restrict E to the subspace $U_\xi^\perp$. It induces a nondegenerate skew-symmetric pairing on $U_\xi^\perp / U_\xi$. Use this as the polarization λ_ξ on $U_\xi^\perp / U_\xi$. Denote the type by δ'. Denote $\mathscr{A}_{g', \delta'}$ by $\overline{F}_\xi$. Therefore over $\overline{F}_\xi$, we have the universal family $\mathcal{A} \times_{\overline{F}_\xi} \mathcal{A}^t$ and constant sheaves $\underline{X}_\xi$ and $\underline{Y}_\xi$.

From now on, choose a 0-cusp with associated maximal isotropic subspace $U \supset U_\xi$. Extend the basis of X_ξ^* to a basis of $U \cap \Lambda$. Extend $\{u'_1, \ldots, u'_r\}$ to $\{u'_1, \ldots, u'_g\}$ such that it is a lift of the basis of $Y = \Lambda / U \cap \Lambda$. Denote the lattice generated by $u'_{r+1}, \ldots, u'_g$ by Y'. Define the integral skew-symmetric

$g \times g$-matrix S as before, and write it in blocks

$$\begin{pmatrix} S_\xi & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

Denote the type of $\phi : Y_\xi \to X_\xi$ by $\mathfrak{d}_1$ and the type of $\phi : Y \to X$ by $\mathfrak{d}$ such that

$$\mathfrak{d} = \begin{pmatrix} \mathfrak{d}_1 & 0 \\ 0 & \mathfrak{d}_2 \end{pmatrix}.$$

Consider the sheaves $\mathscr{H}\mathrm{om}(\underline{X}_\xi, \mathcal{A}^t)$ and $\mathscr{H}\mathrm{om}(\underline{Y}_\xi, \mathcal{A})$. The bundle $\overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$ is the subset of $\mathscr{H}\mathrm{om}(\underline{X}_\xi, \mathcal{A}^t) \times \mathscr{H}\mathrm{om}(\underline{Y}_\xi, \mathcal{A})$ such that the following diagram commutes

$$\begin{tikzcd} Y_\xi \arrow[r, "\phi"] \arrow[d, "c^t"'] & X_\xi \arrow[d, "c"] \\ \mathcal{A} \arrow[r, "\lambda_\xi"] & \mathcal{A}^t \end{tikzcd}.$$

Introduce an automorphism ι on the set of the data (c^t, c).

$$c^t(\lambda) \longmapsto \exp(-\pi i(x'_1(\lambda), \ldots, x'_r(\lambda))S_2\mathfrak{d}_2^{-1})c^t(\lambda) \quad \forall \lambda \in Y_\xi, \tag{4.22}$$

$$c(\alpha) \longmapsto \exp(-\pi i S_3 \mathfrak{d}_1^{-1}(v_1(\alpha), \ldots, v_r(\alpha))^T)c(\alpha) \quad \forall \alpha \in X_\xi. \tag{4.23}$$

Let's explain the notations here. Recall over $\mathbf{C}$, $\mathcal{A}$ can be regarded as a quotient of $\mathbb{G}_m^{g'}$, $g' = g - r$, by the periods Y'. In Equation (4.22), the row vector $\exp(-\pi i(x'_1(\lambda), \ldots, x'_r(\lambda))S_2\mathfrak{d}_2^{-1})$, as an element of $\mathbb{G}_m^{g'}$ is acting on the sections of $\mathcal{A}$. In general, notice that d is invertible in the base ring k. The row vector is an element of $\mu_{2d_{r+1}} \times \ldots \times \mu_{2d_g}$. The choice of the 0-cusp determines a maximal isotropic subgroup for every finite subgroup scheme of

$\mathcal{A}$, and thus a morphism of $\mu_{2d_{r+1}} \times \ldots \times \mu_{2d_g}$ into $\mathcal{A}^2$. In Equation (4.23), the column vector $\exp(-\pi i S_3 \mathfrak{d}_1^{-1}(v_1(\alpha), \ldots, v_r(\alpha))^T)$ should be regarded as an element in $\mu_M \times \ldots \times \mu_M$, acting on $\mathcal{A}^t$. The morphism of $\mu_M \times \ldots \times \mu_M$ into $\mathcal{A}^t$ is determined by the choice of the 0-cusp and the polarizations. The map ι is an automorphism on the set of (c^t, c) that makes the diagram commute, because $S_2 = -S_3^T$ and they are both integral matrices.

Therefore, over $\overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$ there is an automorphism ι^* of the tautological extensions

$$1 \longrightarrow T \longrightarrow \widetilde{G} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0,$$

$$1 \longrightarrow T^t \longrightarrow \widetilde{G}^t \xrightarrow{\pi^t} \mathcal{A}^t \longrightarrow 0.$$

Use the extensions after composing ι^*, and denote them by $\widetilde{G}$ and $\widetilde{G}^t$. There is a polarization $\widetilde{G} \to \widetilde{G}^t$. Let $\mathbb{L}_\xi^* \subset \Gamma^2 U_\xi$ be the integral structure from the integral polarized tropical abelian varieties, and $\mathbb{L}_\xi \subset S^2 U_\xi^*$ be its dual. Identify $\mathcal{C}(F_\xi)$ with $\mathcal{C}(X_\xi)$, and write the torus as $T_\xi = \mathbb{L}_\xi^* \otimes \mathbb{G}_m$. The T_ξ-bundle Ξ_ξ over $\overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$ is defined as follows. For any character $\phi(y) \otimes \alpha \in \mathbb{L}_\xi \subset S^2 U_\xi^*$, the push-out along $\phi(y) \otimes \alpha : T_\xi \to \mathbb{G}_m$ is defined to be the rigidified $\mathbb{G}_m$-torsor $(c^t(y) \times c(\alpha))^* \mathcal{P}_A^{-1}$ over $\overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$. Here $\mathcal{P}_A$ is the pull back of the Poincaré bundle $\mathcal{P}_A$ over $\mathcal{A} \times_{\overline{F}_\xi} \mathcal{A}^t$. Since $(c^t(\lambda) \times c(\alpha))^* \mathcal{P}_A^{-1}$ is defined to be the push-out, the pullback of it over Ξ_ξ is canonically trivial. Denote this tautological trivialization by τ_Ξ. The space Ξ_ξ is the moduli space of the trivialization of biextensions $(c^t \times c)^* \mathcal{P}_A^{-1}$. Since $(\mathrm{Id} \times \phi)^* b'$ is symmetric on $Y_\xi \times Y_\xi$, b' is acting

[2]For example, we can choose a level structure of $\mathcal{A}$ that is compatible with the 0-cusp.

on the trivializations of trivial biextensions over $Y_\xi \times X_\xi$. Define τ to be

$$\tau(y, \alpha) = b'(y, \alpha)\tau_\Xi(y, \alpha).$$

Étale locally, we can choose a line bundle $\mathcal{M}$ over $\mathcal{A}$ which gives the polarization λ_ξ. This gives a T-linearized sheaf $\widetilde{\mathcal{L}} = \pi^*\mathcal{M}$ over $\widetilde{G}$. Moreover, there exists a trivialization ψ compatible with the universal trivialization τ étale locally. We make these étale base changes so that ψ and $\mathcal{M}$ are defined over Ξ_ξ.

The basic data is the second Voronoi fan $\Sigma(X_\xi)$ with the support $\mathcal{C}(X_\xi)^{\mathrm{rc}}$, and the sections $\varphi_{\mathscr{P}}$ for each $C(\mathscr{P}) \in \Sigma(X_\xi)$ that is in the interior of $\mathcal{C}(X_\xi)$. Similar to $\mathbb{L}_{\mathscr{T}}$, $\mathbb{L}_\tau$, we can define $\mathbb{L}_{\mathscr{P}}$ for any polyhedral cone $C(\mathscr{P}) \in \Sigma(X_\xi)$. Assume $I_{\mathscr{P}}$ is the set of all minimal triangulations $\mathscr{T}$ that refines $\mathscr{P}$. Define

$$\mathbb{L}_{\mathscr{P}} := \sum_{\mathscr{T} \in I_{\mathscr{P}}} \mathbb{L}_{\mathscr{T}}$$

Let $T_{\mathscr{P}}$ denote the torus with character group $\mathbb{L}_{\mathscr{P}}$. Étale locally, pick a section of the T_ξ-torsor Ξ_ξ and define the morphism $\Xi_\xi \to T_\xi$. Make the étale base change $\Xi_{T,\mathscr{P}} := \Xi_\xi \times_{T_\xi} T_{\mathscr{P}}$ along the morphisms $T_{\mathscr{P}} \to T_\xi$. Define $\Xi_{\mathscr{P}} := \Xi_{T,\mathscr{P}} \times^{T_{\mathscr{P}}} U_{C(\mathscr{P})}$, and the closed subscheme $\mathfrak{Z}_{\mathscr{P}} = \Xi_{T,\mathscr{P}} \times^{T_{\mathscr{P}}} V(C(\mathscr{P}))$. Define $\widehat{\Xi}_{\mathscr{P}}$ to be the completion of $\Xi_{\mathscr{P}}$ along $\mathfrak{Z}_{\mathscr{P}}$. For each cone $C(\mathscr{P})$, the fiber bundle $\widehat{\Xi}_{\mathscr{P}}$ satisfies the Assumption 4.21, and serves as the formal base along the boundary. While the fiber bundle $\Xi_{\mathscr{P}}$ provides the étale neighborhoods near the boundary. Fix the log structure to be the divisorial log structure, and denote the log schemes by $\Xi^{\dagger}_{\mathscr{P}}$ and $\widehat{\Xi}^{\dagger}_{\mathscr{P}}$.

Remark 4.50. In the complex analytic category, we have the toroidal compactification. Recall that $\mathfrak{D}_\xi := \mathfrak{D}(F_\xi)$ is a $\mathcal{P}'_{\xi,\mathbf{C}}$-principal bundle $(F_\xi \ltimes \mathcal{V}_\xi) \ltimes \mathcal{P}'_{\xi,\mathbf{C}}$. The stack $\overline{F}_\xi$ corresponds to the quotient of F_ξ by an arithmetic group G_h, and $\overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$ corresponds to the quotient of $F_\xi \ltimes \mathcal{V}_\xi$ by $G_h \ltimes V$.

To extend the families, it suffices to consider the cones of maximal dimension $C(\mathscr{T}) \in \Sigma(X_\xi)$, because $\{U_{C(\mathscr{T})}\}$ is an open covering of $X_{\Sigma,\xi}$. Consider the base $\widehat{\Xi}_{\mathscr{T}}$. The function $\varphi_{\mathscr{T}} : X_{\xi,\mathbf{R}} \to S^2U^*_\xi$, with the log structure on $\widehat{\Xi}^{\dagger}_{\mathscr{T}}$, is compatible with the degeneration data τ, ψ. By Theorem 4.34, we have a family $\pi : (\mathcal{X}_Z, P_Z, \mathcal{L}_Z, \varrho_Z) \to (Z, M_Z)$ for each admissible closed subscheme $Z \subset \Xi_{\mathscr{T}}$, such that $(\mathcal{X}_Z, P_Z, \mathcal{L}_Z, G_Z, \varrho_Z) \to (Z, M_Z)$ is an object in $\overline{\mathscr{T}}_{g,d}(Z)$. Moreover, the construction is functorial, so we can get algebraic families $(\mathcal{X}_{\mathscr{P}}, \mathcal{L}_{\mathscr{P}}, G_{\mathscr{P}}, P_{\mathscr{P}}, \varrho_{\mathscr{P}})$ over the formal bases $\widehat{\Xi}_{\mathscr{P}}$ by Grothendieck's existence theorem ([GD63] EGA III_1 5.4.5). The polarization of the generic fiber is of type δ. It can be verified by checking the polarization type of the complex points. To simplify the notations and to follow the constructions in [FC90], when $\bar{\sigma} = C(\mathscr{P})$ is a cone in $\Sigma(F_\xi)$, we also denote the formal base $\widehat{\Xi}_{\mathscr{P}}$ by S_σ, and the family by $\pi : (\mathcal{X}_\sigma, \mathcal{L}_\sigma, G_\sigma, P_\sigma, \varrho_\sigma)/(S_\sigma, M_\sigma)$. We call this miniversal family a good formal σ-model.

Remark 4.51. By ([FC90] Chap. I Proposition 2.7), the semi-abelian group scheme $G_{\mathscr{P}}$ is unique up to a unique isomorphism, and τ can be intrinsically defined by $G_{\mathscr{P}}$. Therefore the definition of τ does not depend on the choice of S_ξ, the 0-cusp, and the matrix S. Moreover, the isomorphism class of the family $(\mathcal{X}_{\mathscr{P}}, \mathcal{L}_{\mathscr{P}}, G_{\mathscr{P}}, P_{\mathscr{P}}, \varrho_{\mathscr{P}})$ does not depend on the choices of $\psi, \mathcal{M}$. The choices of $\psi, \mathcal{M}$ are part of the data called a framing in ([Ale02] Definition 5.3.7). It is possible that, if we start from different data, $\mathcal{L}_{\mathscr{P}}$ is changed to $\mathcal{L}_{\mathscr{P}} \otimes \pi^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ over the base $\widehat{\Xi}_{\mathscr{P}}$. We allow this in our definition of an isomorphism of families.

After a finite base change if necessary, the order of $G_{\mathscr{T},\xi}$ would be invertible in k. In this case, $G_{\mathscr{T},\xi}$ is étale, and the étale pre-equivalence relation can be described explicitly as follows. Denote $U_\xi := \coprod U_\alpha$. Let $U_\alpha = U_{C(\mathscr{T}_i)}$ and $U_\beta = U_{C(\mathscr{T}_2)}$. If $\alpha \neq \beta$, use the étale morphism $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2} : U_\tau \to U_\alpha \times U_\beta$. For each α such that $\sigma_\alpha = C(\mathscr{T})$, use the automorphism group $G_{\mathscr{T},\xi}$ with the Cartier dual $\mathbb{L}_{\mathscr{T}}/\mathbb{L}_\xi$. Let $R_\xi \rightrightarrows U_\xi$ denote the étale pre-equivalence relation generated by $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2}$ and the group $G_{\mathscr{T},\xi}$. Denote the algebraic stack $[U_\xi/R_\xi]$ by $X_{\Sigma,\xi}$.

The étale pre-equivalence relation $R_\xi \rightrightarrows U_\xi$ induces an étale pre-equivalence relation $R\Xi_\xi \rightrightarrows U\Xi_\xi$ on $U\Xi_\xi := \coprod \Xi_{\mathscr{P}}$. Define $\overline{\Xi}_\xi = [U\Xi_\xi/R\Xi_\xi]$. It has a stratification parametrized by $\Sigma(X_\xi)$. Let $\mathcal{O}_\xi$ be the union of all $\mathfrak{Z}_{\mathscr{P}}$ such that $\mathscr{P}$ is X_ξ-periodic bounded paving, i.e. $C(\mathscr{P})$ is inside the interior $\mathcal{C}(X_\xi)^\circ$. The formal completion $\widehat{\overline{\Xi}}_\xi$ of $\overline{\Xi}_\xi$ along $\mathcal{O}_\xi$ is the base over the cusp F_ξ.

Again we are free to make an étale base change without changing the notations $\overline{\Xi}_\xi$ and $\widehat{\overline{\Xi}}_\xi$, and $\Xi_{\mathscr{P}}$. The pull back of the divisorial log structure is still divisorial log structure under étale base change. Over $\Xi_{\mathscr{P}}$, the charts are given by the monoid $P_{\mathscr{P}}$.

Recall we have the line bundle $\mathcal{L}_q$ on $X_{\Sigma,\xi}$. Let the $\mathbb{G}_m$-torsor be $\widetilde{\mathbb{L}}_q$. Do the contracted product of $\widetilde{\mathbb{L}}_q$ with the trivial $\mathbb{G}_m$-torsor over Ξ_ξ, we get a $\mathbb{G}_m$-torsor over $\overline{\Xi}_\xi$. The corresponding line bundle over $\overline{\Xi}_\xi$ and its restrictions are still denoted by $\mathcal{L}_q$.

Let Z be a closed subscheme of $\overline{\Xi}_\xi$ whose image is contained in $\mathcal{O}_\xi$, and does not intersect the closed strata corresponding to the cones in the boundary of $\mathcal{C}(X_\xi)$. Let the ideal sheaf over the small étale site be $\mathscr{I}_Z$. To simplify the notations, if $U \to \overline{\Xi}_\xi$ is étale, $U \times_{\overline{\Xi}_\xi} Z$ is denoted by $Z \cap U$. We generalize Theorem 4.46,

Theorem 4.52. *For any such Z, there exists a flat proper family $\pi : (\mathcal{X}_Z, P_Z, \mathcal{L}_Z, G_Z, \varrho_Z) \to (Z, M_Z)$, with the following properties:*

1. *Restrict to each toric $\Xi_{\mathscr{P}}$, it is canonically isomorphic to a family constructed above.*
2. *The family $\pi : (\mathcal{X}_Z, P_Z, \mathcal{L}_Z, G_Z, \varrho_Z) \to (Z, M_Z)$ is an object in $\overline{\mathscr{T}}_{g,d}(Z)$.*
3. *If $Z' \to Z$ is closed embedding, then the family $(\mathcal{X}_{Z'}, P_{Z'}, \mathcal{L}_{Z'}, G_{Z'}, \varrho_{Z'})$ is a pull back of $(\mathcal{X}_Z, P_Z, \mathcal{L}_Z, G_Z, \varrho_Z)$.*
4. *$\pi_*\mathcal{L}^m$ is locally free, and $R^i\pi_*\mathcal{L}^m = 0$, for each $m > 0, i > 0$.*
5. *Pick $\vartheta_{A,\omega} \in \mathrm{H}^0(A, \mathcal{M}_\omega)$ for each $\omega \in B(\mathbf{Z})$, such that $\vartheta_{A,\omega}$ does not vanish along any fiber A_x. Then over étale neighborhood Z_α[3], we have a sec-*

[3] It will be defined in the proof.

tion ϑ of $\pi_*\mathcal{L}$, defining an effective divisor Θ_α such that $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, G_\alpha, \Theta_\alpha, \varrho_\alpha)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}(Z_\alpha)$. However, Θ_α does not descend to Z.

Proof. The proof is similar to the proof of Theorem 4.46. For each maximal cone $\sigma_\alpha \subset \mathcal{C}(X_\xi)$ that corresponds to a minimal triangulation $\mathscr{T}$, let $\Xi_{\mathscr{T}}$ be denoted by U_α. The collection $\underline{\mathscr{U}} = \{U_\alpha\}$ is an étale open covering of $\overline{\Xi}_\xi$. Pull back Z to U_α, and denote $Z \cap \Xi_\xi(\alpha)$ by Z_α. Because Z doesn't intersect the closed strata corresponding to the cones in the boundary of $\mathcal{C}(X_\xi)$, Z_α is admissible for $\varphi_{\mathscr{T}}$. We can check this on an affine base $S = \operatorname{Spec} R$ where Ξ_ξ is trivial as a T_ξ-torsor. The argument is the same as that in Theorem 4.46. Define the sheaf of $\mathcal{O}_A$-algebra $\mathcal{R}_\alpha$ for each Z_α, and $\widetilde{\mathcal{X}}_\alpha = \mathbf{Proj}\,\mathcal{R}_\alpha$. We have the family $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, P_\alpha, G_\alpha, \varrho_\alpha) \to (Z_\alpha, M_\alpha)$.

Consider the group action of $G_{\mathscr{T},\xi}$ on $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, P_\alpha, G_\alpha, \varrho_\alpha) \to (Z_\alpha, M_\alpha)$. Let ζ be an S-point of $G_{\mathscr{T},\xi}$ for some k-scheme S. Choose a local trivialization of Z_α. The element ζ is acting on $\Xi_{\mathscr{T}}$ by the morphism $\Xi_{\mathscr{T}} \times S \to \operatorname{Spec} \mathcal{O}_S[P_{\mathscr{T}}]$, and

$$g : \mathrm{X}^p \mapsto \mathrm{X}^p(\zeta)\mathrm{X}^p.$$

This action is independent of the choice of the trivialization because the torus action is commutative.

Regard $S(Q_{\varphi_{\mathscr{T}}})$ as a submonoid of $\mathbb{X} \times \mathbb{L}_{\mathscr{T}}$. The pullback $(g^*\widetilde{\mathcal{X}}_\alpha, g^*\widetilde{\mathcal{L}}_\alpha)$

is isomorphic to $(\widetilde{\mathcal{X}}_\alpha, \widetilde{\mathcal{L}}_\alpha)$ by an isomorphism

$$F_\zeta : g^*\mathcal{R}_{\mathscr{T}} \longrightarrow \mathcal{R}_{\mathscr{T}},$$
$$1 \otimes \mathrm{X}^p \otimes \mathcal{O}_\omega \otimes \mathcal{M}^m\theta^m \longmapsto \mathrm{X}^p(\zeta)B(-(m,\omega),\zeta)\mathrm{X}^p \otimes \mathcal{O}_\omega \otimes \mathcal{M}^d\theta^d.$$

By the same argument as in Theorem 4.46, F_ζ induces an isomorphism between the families $(g^*\mathcal{X}_\alpha, g^*\mathcal{L}_\alpha, g^*P_\alpha, g^*\varrho_\alpha) \to (g^*Z_\alpha, g^*M_\alpha)$ and $(\mathcal{X}_\alpha, \mathcal{L}_\alpha, P_\alpha, \varrho_\alpha) \to (Z_\alpha, M_\alpha)$.

Consider the intersection $U_\tau = U_\alpha \times_{\Xi_\xi} U_\beta$. Glue $(\widetilde{\mathcal{X}}_\alpha, \widetilde{\mathcal{L}}_\alpha)$ and $(\widetilde{\mathcal{X}}_\beta, \widetilde{\mathcal{L}}_\beta)$ by using the monoid of line bundles $\oplus\mathcal{L}_q$. That is, choose an étale covering where the torsor Ξ_ξ is trivial. Then choose a trivialization, and define a map $\varphi_{\alpha\beta}$ over $Z_\tau := Z \cap U_\tau$.

$$\varphi_{\alpha\beta} : \mathcal{R}_\alpha \longrightarrow \mathcal{R}_\beta$$
$$\mathrm{X}^p_\alpha \otimes \mathcal{O}_\omega \otimes \mathcal{M}^m\theta^m \longmapsto \mathrm{X}_\beta^{p+mg^{\alpha\beta}(\omega)} \otimes \mathcal{O}_\omega \otimes \mathcal{M}^m\theta^m.$$

Here we use the model $S(X) \rtimes_{\varphi_{\mathscr{T}}} P_{\mathscr{T}}$ instead of $S(Q_{\varphi_{\mathscr{T}}})$. Since the difference $mg^{\alpha\beta}(\omega)$ does not depend on p, the twist by $\oplus\mathcal{L}_q$ does not depend on the trivialization of the T_ξ-torsor. Moreover, $\varphi_{\alpha\beta}$ is an isomorphism of algebras just as in the GKZ case. Moreover, $\varphi_{\alpha\beta}$ commutes with the action of Y as in Theorem 4.46. Therefore, we can glue $(\mathcal{X}_\alpha, \mathcal{L}_\alpha)$ and $(\mathcal{X}_\beta, \mathcal{L}_\beta)$. Together with $\{F_\zeta\}$, we get $(\mathcal{X}_Z, \mathcal{L}_Z)$ over Z.

By Mumford's construction in ([FC90] Chap. III), we also have a semi-abelian scheme G_Z over Z.

Since Z has the support inside the boundary $\mathcal{O}_\xi$, the action of $\widetilde{G}$ on the relative complete model $\widetilde{\mathcal{X}}_{Z_\alpha}$ is defined in ([Ols08] 4.1.14). For each S-point $g \in \widetilde{G}(S)$ that corresponds to a point $a \in \mathcal{A}(S)$ and a compatible isomorphism $\iota_g : t_a^*\mathcal{O}_{-\omega} \to \mathcal{O}_{-\omega}$, the action $g^* : T_a^*\mathcal{R} \to \mathcal{R}$ is defined locally as follows. Choose a local trivialization of $T_a^*\mathcal{M} \otimes \mathcal{M}^{-1}$, the isomorphism g^* is the one induced by ι_g. Therefore it commutes with the twist by $\oplus\mathcal{L}_q$. This means we can glue the actions $\{\varrho_{Z_\alpha}\}$, and get an action of G_Z on $\mathcal{X}_Z$, denoted by ϱ_Z.

Since the log structure is defined étale locally, we can apply the same argument as in the proof of Theorem 4.46. As a result, we get the log structure P_Z on $\mathcal{X}_Z$, and M_Z on Z. Now 1) and 2) are proved.

For each $\omega \in B(\mathbf{Z}) = X/\phi(Y)$, choose $\vartheta_{A,\omega} \in \mathrm{H}^0(A, \mathcal{M}_\omega)$ such that $\vartheta_{A,\omega}$ does not vanish along each fiber A_x. Define ϑ as the descent (Y-action) from

$$\tilde{\vartheta} := \sum_{\omega \in B(\mathbf{Z})} \sum_{\lambda \in Y} S_\lambda^* \vartheta_{A,\omega}.$$

Let Θ be the zero locus of ϑ. This is well defined over Z_α for each α, and is denoted by Θ_α. Then $(\mathcal{X}_\alpha, \Theta_\alpha, \mathcal{L}_\alpha, \varrho_\alpha)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{d,g}(Z_\alpha)$. Since $g^{12} = 0$ for $\omega \in X_\xi$, the divisors Θ_α are preserved by the étale relation $p_{\tau,\mathscr{T}_1} \times p_{\tau,\mathscr{T}_2}$. However, they are not preserved by the group actions $G_{\mathscr{T}}$. As in Theorem 4.46, under the action of $g \in G_{\mathscr{T}}$, each component $\sum_{\lambda \in Y} S_\lambda^* \vartheta_{A,\omega}$ will be changed by a constant in μ_M, and this constant depends on ω. Therefore Θ_α only exists étale locally, and does not descend to Z. This is the statement 5).

Since the twists by $\oplus\mathcal{L}_q$ commute with the base change, we have 3). Finally 4) follows from Corollary 4.18. $\square$

For every closed subscheme of $\widehat{\Xi}_\xi$, we have assigned an object in $\overline{\mathscr{T}}_{g,d}$. By Theorem 4.52 3), this assignment is functorial. Moreover, the pro-family is effective over each complete local ring. Therefore, it defines a family in $\overline{\mathscr{T}}_{g,d}$ over the formal stack $\widehat{\Xi}_\xi$. Let it be denoted by $((\mathcal{X}_\xi^f)^\dagger, \mathcal{L}_\xi^f, G_\xi^f, \varrho_\xi)$ over $\widehat{\Xi}_\xi^\dagger$.

Proposition 4.53. *The family* $((\mathcal{X}_\xi^f)^\dagger, \mathcal{L}_\xi^f, G_\xi^f, \varrho_\xi) \to \widehat{\Xi}_\xi^\dagger$ *is* $\overline{P}(F_\xi)$*-equivariant.*

Proof. The piecewise affine sections φ commute with the action of $\overline{P}(F_\xi)$. $\square$

Remark 4.54. Ideally, we would like to define the moduli space over $k = \mathbf{Z}[1/d]$. Construct the same bases $\overline{\Xi}_\xi$ (resp. $\widehat{\Xi}_\xi$, $\Xi_{\mathscr{P}}$) by the same construction over $\mathbf{Z}[1/d]$, without applying the automorphism ι^*. Therefore $\overline{\Xi}_\xi$ (resp. $\widehat{\Xi}_\xi$, $\Xi_{\mathscr{P}}$) over $\mathbf{Z}[1/d, \zeta_M]$ is the base change of $\overline{\Xi}_\xi$ (resp. $\widehat{\Xi}_\xi$, $\Xi_{\mathscr{P}}$) over $\mathbf{Z}[1/d]$. However, the construction of the stack really requires the families over the formal bases. Since $\mathbf{Z}[1/d] \to \mathbf{Z}[1/d, \zeta_M]$ is a faithfully flat, the base change for $\overline{\Xi}_\xi$ (resp. $\widehat{\Xi}_\xi, \Xi_{\mathscr{P}}$) is also faithfully flat. It is possible to construct the miniversal families by descent. Or equivalently, we hope to construct the degeneration data (c^t, c, τ, ψ) from a faithfully flat descent.

4.4.3 Algebraization and Gluing

After we have obtained the miniversal families over the formal bases (complete rings), we need to extend them to étale bases (rings of finite type over k). The étale neighborhoods of the boundary in $\Xi_{\mathscr{P}}$ server as the étale bases.

We follow the procedure in [FC90]. Let σ denote the rational cone $C(\mathscr{P})$ associated with a bounded paving $\mathscr{P}$. Since $\Xi_{\mathscr{P}}$ is a fibered by the toric variety U_σ, $\Xi_{\mathscr{P}}$ has the natural stratification induced by the toric stratification of U_σ. Then we can define the étale constructible sheaf $\underline{X}_\xi$ (resp. $\underline{Y}_\xi$). If τ is a face of σ, the elements in τ are quadratic forms over $X_{\tau,\mathbf{R}}$, and X_τ (resp. Y_τ) is the quotient of X_σ (resp. Y_σ). Then, over the τ-stratum, define the sheaf $\underline{X}_\xi$ (resp. $\underline{Y}_\xi$) to be the constant sheaf X_τ (resp. Y_τ). Moreover, over each τ-stratum, we have the tautological bilinear pairing $B : Y_\tau \times X_\tau \to \mathbb{L}_\tau$. The elements in $\mathbb{L}_\tau$ are sections of $\mathcal{K}^*/\mathcal{O}^*$ over the toric variety $U_\tau \subset U_\sigma$. Therefore, we get a pairing $\underline{B}_\xi : \underline{Y}_\xi \times \underline{X}_\xi \to \underline{\mathrm{Div}}\Xi_{\mathscr{P}}$ from the toric data.

If $(\mathcal{X}_\sigma, \mathcal{L}_\sigma, G_\sigma, P_\sigma, \varrho_\sigma)$ over (S_σ, M_σ) is a good formal σ-model, we can forget about the other data, and get a good formal σ-model (G_σ, λ) over S_σ, with a general polarization λ ([FC90] Chap. IV Definition 3.2 & Proposition 3.3 (i)). Notice that all the results in ([FC90] Chap. III Sect. 9 & 10) are proved for general polarizations, therefore (loc. cit. Chap. IV Definition 3.2 & Proposition 3.3 (i)) can be naturally generalized for the general polarizations. Since (S_σ, M_σ) is the divisorial log structure for the toroidal boundary ∂S_σ, let the log differential $\Omega^1_{(S_\sigma, M_\sigma)}$ be denoted by $\Omega^1_{S_\sigma}[\,\mathrm{d}\log\infty]$, the differential with log poles along the boundary. In particular for good formal σ-model, we have

Proposition 4.55. *Suppose $(\mathcal{X}_\sigma, \mathcal{L}_\sigma, G_\sigma, P_\sigma, \varrho_\sigma)$ over (S_σ, M_σ) is a good formal σ-model.*

1. *The sheaves and the pairing* $(\underline{B}, \underline{X}, \underline{Y})$ *obtained from the polarized semi-abelian scheme* (G_σ, λ) *agree with the sheaves and the pairing* $(\underline{B}_\xi, \underline{X}_\xi, \underline{Y}_\xi)$ *obtained from the toric variety fibers.*

2. *Let* $\Omega = \underline{\Omega}(G_\sigma/S_\sigma)$ *be the dual of the relative invariant Lie algebra of* G_σ *and* $\Omega^t = \underline{\Omega}(G^t_\sigma/S_\sigma)$ *be the dual of the relative invariant Lie algebra of* G^t_σ*. The Kodaira–Spencer map induces an isomorphism* $S^2\Omega^t \cong \Omega^1_{S_\sigma}[\,\mathrm{d}\log\infty]$.

Notice that our formal base is the same with that in [FC90], we can do the same approximation to the rings of finite type over $\Xi_{\mathscr{P}}$. The point is the two properties in Proposition 4.55 are preserved in the process of approximation. Therefore we have (loc. cit. IV Proposition 4.3, 4.4) for our case.

Proposition 4.56. *Let* $\sigma = C(\mathscr{P})$*,* R *be the strict local ring of a geometric point* $\overline{x}$ *of the* σ*-stratum of* $\Xi_{\mathscr{P}}$*,* I *be the ideal defining the* σ*-stratum, and* $\widehat{R}$ *be the* I*-adic completion of* R*. There exists an étale neighborhood* $S' = \operatorname{Spec} R'$ *of* $\overline{x}$*, a family* $(\mathcal{X}, \mathcal{L}, P, G, \varrho)$ *over* (S', M')*, and an embedding* $R' \to \widehat{R}$ *close to the canonical inclusion, such that*

1. *The family* $(\mathcal{X}, \mathcal{L}, P, G, \varrho)$ *is an object in* $\overline{\mathscr{T}}_{g,d}(S')$*. The log structure is the divisorial log structure.*

2. *The map* $\operatorname{Spf} \widehat{R} \to \widehat{\Xi}_{\mathscr{P}}$ *coincides on the* σ*-stratum with the map induced by the inclusion* $R' \to \widehat{R}$*.*

3. *Over R/I, $(\mathcal{X}, \mathcal{L}, P, G, \varrho)$ is isomorphic to the pull back of the good formal σ-model $(\mathcal{X}_\sigma, \mathcal{L}_\sigma, P_\sigma, G_\sigma, \varrho_\sigma)$.*

4. *The étale sheaves and the pairing $(\underline{B}, \underline{X}, \underline{Y})$ obtained from the family G coincide with the pull backs of the étale sheaves and the pairing $(\underline{B}_\xi, \underline{X}_\xi, \underline{Y}_\xi)$ over $\Xi_{\mathscr{P}}$.*

5. *The Kodaira-Spencer map induces an isomorphism $S^2\Omega^t \cong \Omega^1_{R'}[\,\mathrm{d}\log\infty]$.*

Proof. For (2)–(5), the same proof as that of ([FC90] Chap. IV Proposition 4.4). The statement (1) follows from (3) and ([Ols08] Theorem 5.9.1). □

We call the family obtained above a good algebraic σ-model. For any geometric point $\bar{x}$ over $x \in S$ in the τ-stratum of a good algebraic σ-model over S, denote the strict henselization by $\widehat{R}_{\bar{x}}$. The polarized family G over $\widehat{R}_{\bar{x}}$ gives the degeneration data. By Proposition 4.56 (4), we know $\underline{B}$ agrees with $\underline{B}_\xi|_\tau$. By the universal property described in (at the end of the third paragraph [FC90] p. 106), this defines a morphism $f : \mathrm{Spf}\,\widehat{R}_{\bar{x}} \to \Xi_\tau$. The family G over $\widehat{R}_{\bar{x}}$ is a good formal τ-model. The boundary divisors agree under f. It follows that f is strict for the log structures. Moreover, by Proposition 4.56 (5), the log differentials $\Omega^1_{(S,M)} \cong f^*\Omega^1_{(\Xi_\tau, M_\tau)}$. Therefore, f is log étale. Openness of versality follows from the following lemma.

Lemma 4.57. *If a log morphism $f : (X, M_X) \to (Y, M_Y)$ is strict, then f being formally log étale implies that the underlying morphism f between schemes is formally étale.*

Proof. Consider the affine thickening diagram

$$\begin{tikzcd} T_0 \arrow[r, "a^0"] \arrow[d, hook, "j"'] & X \arrow[d, "f"] \\ T \arrow[r, "a"] & Y \end{tikzcd},$$

where T is affine, and j is a closed immersion defined by a nilpotent ideal. Provide T (resp. T_0) the log structure a^*M_Y (resp. $a_0^*M_X$), the diagram is still commutative. Because f is strict, j is strict. Since f is formally log étale, there is a unique lift $b : T \to X$. Thus f is formally étale. □

The good algebraic models are the étale neighborhoods of the compactification. Define U to be the disjoint union of finitely many good algebraic models that cover all strata $\mathfrak{Z}_{\mathscr{P}}$ up to the action of $\Gamma(\delta)$. This is possible because there are only finitely map 0-cusps up to $\Gamma(\delta)$-action, and for each 0-cusp F, the number of $\overline{P}(F)$-orbits of cones in $\Sigma(F)$ is finite. The cone $\{0\}$ corresponds to the moduli space $\mathscr{A}_{g,\delta}$. However, by ([Ols08] Proposition 5.1.4), the universal family is interpretated as the family $(\mathcal{X}, \mathcal{L}, \varrho, G)$ with G abelian. Let U_0 denote the dense open stratum of U where G is abelian. Let R_0 denote the étale relation $R_0 := U_0 \times_{\mathscr{A}_{g,\delta}} U_0$. Define R to be the normalization of the image $R_0 \to U \times U$, i.e. normalization of the closure Z of the image of R_0 in $U \times U$ with respect to the finite extensions of the function fields at the maximal points of Z. We can extend the isomorphisms by an explicit construction. To save the work, we use the following analog of ([FC90] Chap. I Proposition 2.7).

Proposition 4.58. *Let S be a noetherian normal scheme, and $\pi_i : (\mathcal{X}_i, P_i, \mathcal{L}_i, G_i, \varrho_i)$ for $i = 1, 2$ be two good algebraic models over S. Suppose that over a dense open subscheme U of S, the log structures P_i are trivial, and there is an isomorphism f_U between $(\mathcal{X}_i, P_i, \mathcal{L}_i, G_i, \varrho_i)|_U$. Then f_U extends to a unique isomorphism $f : (\mathcal{X}_1, P_1, G_1, \varrho_1) \to (\mathcal{X}_2, P_2, G_2, \varrho_2)$ and there exists a line bundle $\mathcal{M}$ over S, such that $\mathcal{M}|_U$ is trivial, and $\mathcal{L}_1 \otimes \pi_1^* \mathcal{M} \cong f^* \mathcal{L}_2$.*

Proof. For the extension problem of morphisms between data $(\mathcal{X}_i, G_i, \varrho_i)$, by a standard reduction process ([FC90] Chap. I Proof of Proposition 2.7 Step (a)), it suffices to consider the case when S is the spectrum of a discrete valuation ring. This case is proved by ([Ols08] Proposition 5.11.6). Let the unique extension be f. From this reduction to discrete valuation rings, we also know that $f^* \mathcal{L}_2$ and $\mathcal{L}_1$ are isomorphic on each fiber. Since $\mathcal{X}_1$ is projective over S and the fibers are all integral, by ([Har77] Chap. III Exercise 12.4), there exists $\mathcal{M}$ over S such that $\mathcal{L}_1 \otimes \pi_1^* \mathcal{M} \cong f^* \mathcal{L}_2$. Finally, by ([Ols08] Proposition 5.10.2), f preserves the log structures. □

By Proposition 4.58, the isomorphism between the universal families over R_0 is extended to the isomorphism between the good algebraic models over R[4]. Since morphisms in $\overline{\mathscr{A}}_{g,d}$ is defined up to an action of $\mathbb{G}_m$, we have

Corollary 4.59. Let $R' := U \times_{\overline{\mathscr{A}}_{g,d}} U$. We have defined a morphism $R \to R'$, and this map is injective.

[4]Again we say f preserves the line bundles if $\mathcal{L}_1 \otimes \pi_1^* \mathcal{M} \cong f^* \mathcal{L}_2$.

Denote the two projections $R \to U$ by s and t. Notice that the definitions of U and R only involve the collection of fans $\widetilde{\Sigma}$ and the algebraization of the bases. Therefore we can use the same U and R as those in ([FC90] Chap. IV, Sect. 5.). We simply replace the semi-abelian schemes over U and R by AN families.

Proposition 4.60 ([FC90] Chap.IV Lemma 5.2)**.** *Suppose R is a normal complete local ring which is strictly henselian, K its field of fractions and κ its residue field. Assume furthermore that $G^\dagger$ is a semi-abelian scheme over R whose generic fiber $G^\dagger_K$ is an abelian variety with a polarization λ_K of type δ. Associated with this we have the character group X_ξ (resp. Y_ξ) of the torus part of the special fiber G_s (resp. G^t_s), a polarization $\phi : Y_\xi \to X_\xi$, and a bimultiplicative form b on $Y_\xi \times X_\xi \to K^*$, defined up to units in R, such that $b(Y_\xi, \phi(Y_\xi))$ is symmetric. The following statements are equivalent*

1. *Write X_ξ as the quotient lattice of $X = \mathbf{Z}^g$. If $v : K^* \to \mathbf{Z}$ is any discrete valuation defined by a prime ideal of R of height one, $v \circ b$ defines a positive semi-definite symmetric bilinear form on X. There exists a (closed) cone $C(\mathscr{P})$ in the second Voronoi fan $\Sigma(X)$ which contains all $v \circ b$ obtained this way.*

2. *$(G^\dagger, \lambda)$ is the semi-abelian scheme of the pull-back of a good formal σ-model $(\mathcal{X}_\sigma, \mathcal{L}_\sigma, G_\sigma, \varrho_\sigma)/S_\sigma$ via a morphism $\operatorname{Spec} R \to \Xi_\sigma$ (equivalently, a map $\operatorname{Spf} R \to S_\sigma$) for some (open) cone σ in $\Sigma(X)$.*

The cone $C(\mathscr{P})$ in (1) and the cone σ in (2) are in the same $\mathrm{GL}(X, Y)$-orbits.

Theorem 4.61. *The morphism $(s,t) : R \to U \times U$ defines an étale groupoid in the category of k-schemes. Let $\overline{\mathscr{A}}^m_{g,\delta}$ be the stack $[U/R]$ over k. It is a proper Deligne-Mumford stack, containing $\mathscr{A}_{g,\delta}$ as an open dense substack. It admits a coarse moduli space. Over $\mathbf{C}$, the coarse moduli space is the toroidal compactification $\overline{\mathcal{A}}_\Sigma$.*

Proof. Notice that none of the arguments in ([FC90] Chap. IV Proposition 5.4, Corollary 5.5 & Theorem 5.7 (1)) uses the principal polarization. By ([Ols08] Theorem 1.4.2), the coarse moduli space exists. By the construction, the coarse moduli space over $\mathbf{C}$ is $\overline{\mathcal{A}}_\Sigma$. □

From now on, let $\overline{\mathcal{A}}_\Sigma$ also denote the coarse moduli space of $\overline{\mathscr{A}}^m_{g,\delta}$ over the ring k.

Proposition 4.62. *The algebraic stack $\overline{\mathscr{A}}^m_{g,\delta}$ has a log structure M_Σ such that $(\overline{\mathscr{A}}^m_{g,\delta}, M_\Sigma)$ is log smooth over $k = \mathbf{Z}[1/d, \zeta_M]$.*

Proof. It follows from the description of a good algebraic model in Proposition 4.56. □

Lemma 4.63. *The algebraic space $\overline{\mathcal{A}}_\Sigma$ is normal.*

Proof. Recall that the local chart for the algebraic stack $\overline{\mathscr{A}}^m_{g,\delta}$ is a toric monoid. By ([Ogu06] Chap. I Proposition 3.3.1.), $\overline{\mathscr{A}}^m_{g,\delta}$ is normal. Since $\overline{\mathscr{A}}^m_{g,\delta}$ is a separated Deligne-Mumford stack, étale locally, it is presented as $[U/G]$ for U a scheme étale over the stack, and G a finite group ([AV02] Lemma 2.2.3). Étale locally, the coarse moduli space is the coarse moduli space $[U/G] \to U/G$. Now U is normal, and U/G is a quotient by a finite group, thus is also normal. $\square$

Since d is invertible in k, $\mathscr{A}_{g,d}$ is a disjoint union of $\mathscr{A}_{g,\delta}$ for all polarization type δ of degree d. Correspondingly, $\overline{\mathscr{A}}_{g,d}[1/d]$ is a disjoint union of connected components denoted by $\overline{\mathscr{A}}_{g,\delta}[1/d]$ ([Ols08] 6.2.1). Moreover, $\overline{\mathscr{A}}_{g,\delta}[1/d]$ is log smooth in this case. The coarse moduli space $\overline{\mathcal{A}}_{g,d}$ is a disjoint union of connected components $\overline{\mathcal{A}}_{g,\delta}[1/d]$.

Proposition 4.64. *Over $k = \mathbf{Z}[1/d, \zeta_M]$, the morphism $\overline{F} : \overline{\mathscr{A}}^m_{g,\delta} \to \overline{\mathscr{A}}_{g,\delta}[1/d]$ induced by the AN family is proper, surjective, and representable. The coarse moduli space $\overline{\mathcal{A}}_\Sigma$ is the normalization of $\overline{\mathcal{A}}_{g,\delta}[1/d]$ over k. In particular, over a field of characteristic zero, $\overline{F}$ is an isomorphism.*

Proof. There is a morphism $U \to \overline{\mathscr{A}}_{g,\delta}[1/d]$ induced from Proposition 4.56 (1), and an injection $R \to R'$. This induces a morphism between stacks $[U/R] \to [U/R']$. By ([LMB00] Proposition (3.8)), $[U/R']$ is a substack of $\overline{\mathscr{A}}_{g,\delta}[1/d]$. The composition is a morphism $\overline{F} : \overline{\mathscr{A}}^m_{g,\delta} \to \overline{\mathscr{A}}_{g,\delta}[1/d]$. Since both $\overline{\mathscr{A}}^m_{g,\delta}$ and $\overline{\mathscr{A}}_{g,\delta}[1/d]$ are proper, by ([Ols13] Proposition 10.1.4 (iv)), $\overline{F}$ is proper in the sense of ([Ols13] Definition 10.1.3). The morphism $\overline{F}$ is identity on the dense open substack $\mathscr{A}_{g,\delta}$ of $\overline{\mathscr{A}}^m_{g,\delta}$. Then it is surjective onto the closure $\overline{\mathscr{A}}_{g,\delta}[1/d]$.

By Lemma 2.63 and Corollary 4.59, $\overline{F}$ is representable. In particular, by ([Ols13] Proposition 10.1.2), $\overline{F}$ is proper as a representable morphism.

Let the morphism between the coarse moduli spaces be denoted by $f : \overline{\mathcal{A}}_\Sigma \to \overline{\mathcal{A}}_{g,\delta}[1/d]$. As a morphism between proper spaces, f is proper. Moreover, by Lemma 4.65, f is quasi-finite, and thus finite ([GD67] 8.11.1). Now f is also birational, and $\overline{\mathcal{A}}_\Sigma$ is normal by Lemma 4.63, hence f is the normalization of $\overline{\mathcal{A}}_{g,\delta}[1/d]$. Here the normalization of an algebraic space is defined in ([KM99] Appendix N.3).

Over a field of characteristic zero, $\overline{\mathscr{A}}_{g,\delta}[1/d]$ is Deligne–Mumford. Therefore, we have an étale chart U' over $\overline{\mathscr{A}}_{g,\delta}[1/d]$ which is normal. Then the coarse moduli space $\overline{\mathcal{A}}_{g,\delta}[1/d]$, locally a finite group quotient of U', is also normal. Therefore, $f : \overline{\mathcal{A}}_\Sigma \to \overline{\mathcal{A}}_{g,\delta}[1/d]$ is an isomorphism. On the other hand, the pullback of U' along $\overline{F}$ is an étale chart U for $\overline{\mathscr{A}}^m_{g,\delta}$. Since $U \to U'$ is the normalization, we also have $U' \cong U$. Use ([AV02] Lemma 2.2.3) and the fact that the coarse moduli spaces commute with étale base change, we get the following étale local picture. $\overline{F} : [U/G] \to [U/G']$ for finite groups G and G' , such that

1) The coarse moduli spaces $U/G \to U/G'$ are isomorphic,

2) $G \to G'$ is injective,

3) There is a dense open subscheme $V \subset U/G$ over which $\overline{F}$ is isomorphic.

We claim that $G \to G'$ is also surjective. By 1), for any point $x \in U$ and its stabilizer G'_x , we have $G \cdot G'_x = G'$. If $G \to G'$ were not surjective, there would exist x over V such that $G'_x \neq G_x$. This contradicts with 3). Therefore $\overline{F}$ is an isomorphism. $\square$

Lemma 4.65. *The morphism $f : \overline{\mathcal{A}}_\Sigma \to \overline{\mathcal{A}}_{g,\delta}[1/d]$ is quasi-finite.*

Proof. It suffices to check that for any algebraic closed field $\kappa(\bar{x})$ and point $\bar{x}$: $\operatorname{Spec}\kappa(\bar{x}) \to \overline{\mathscr{A}}_{g,d}$, the fiber $\overline{F}^{-1}(\bar{x})$ is discrete, since f is of finite type and thus quasi-compact. So we fix such a point $\bar{x}$ on the boundary of $\overline{\mathscr{A}}_{g,d}$. Forget about the log structure, it represents a polarized stable semiabelic variety $(X, G, \mathcal{L}, \varrho)$ over $\kappa(\bar{x})$. By ([Ols08] 5.3.4), $(X, G, \mathcal{L}, \varrho)$ determines the decomposition $\mathscr{P}$, the subgroup $\phi : Y \to X$, the maps $c : X \to A$ and $c^t : Y \to A^t$. Since the $\mathscr{P}$-stratum in $\overline{\mathcal{A}}_\Sigma$ is a quotient of the interior $\mathfrak{Z}^\circ_{\mathscr{P}}$ by a discrete group, we can consider the inverse image in $\mathfrak{Z}^\circ_{\mathscr{P}}$, and still denote it by $\overline{F}^{-1}(\overline{x})$. More precisely, $\overline{F}^{-1}(\overline{x})$ is contained in the fiber over $(A, A^t, \phi, \lambda, c, c^t) \in \overline{F}_\xi \ltimes \overline{\mathcal{V}}_\xi$. Let's denote the fiber by $V(\mathscr{P})$. It is a torus and is isomorphic to $\mathrm{H}^0(\Delta_{\mathscr{P}}, \widehat{\underline{\mathbb{L}}})$ in [Ale02]. The claim is that there is no positive dimensional sub locus of $V(\mathscr{P})$ where the fibers are all isomorphic as polarized stable semiabelic varieties (SSAV).

Recall the framing for a polarized SSAV defined in ([Ale02] Definition 5.3.6). By ([Ale02] Theorem 5.3.8), the framed polarized SSAV are classified by the groupoid $M^{\mathrm{fr}}[\Delta_{\mathscr{P}}, c, c^t, \mathcal{M}](\kappa(\overline{x}))$ which is equivalent to $[Z^1(\Delta_{\mathscr{P}}, \widehat{\underline{\mathbb{X}}})/C^0(\Delta_{\mathscr{P}}, \widehat{\underline{\mathbb{X}}})]$. Fix the framing, an element in $Z^1(\Delta_{\mathscr{P}}, \widehat{\underline{\mathbb{X}}})$ is the gluing data denoted by (ψ_0, τ_0). Fix $(A, A^t, \phi, \lambda, c, c^t)$, we can construct the unframed groupoid $M[\Delta_{\mathscr{P}}, c, c^t](\kappa(\overline{x}))$, i.e. the groupoid of polarized SSAV as in (loc. cit. 5.4.4). Let $\operatorname{Pic}^\lambda A$ be the component of $\operatorname{Pic} A$ of polarization λ. Over $\operatorname{Pic}^\lambda A$, take the family $Z^1(\Delta_{\mathscr{P}}, \widehat{\underline{\mathbb{X}}})$, and denote the union by M^{fr}. Then quotient out the choice of the framing. We won't write down all the equivalent relations, because it suffices to only consider the choices that are not discrete. There are two type of non-discrete choices we have to quotient out. One is the choice of the projection to A, i.e. a choice of a point in the minimal A-orbit. See (loc. cit. Definition 5.3.7 (4)). The other is the action of $C^0(\Delta_{\mathscr{P}}, \underline{\widehat{\mathbb{X}}})$.

In our construction, we have chosen $\mathcal{M}$ and ψ étale locally. Therefore, there is a natural framing for our family over $V(\mathscr{P})$. Since we construct the base Ξ_ξ as the moduli space for the degeneration data τ. It defines a finite map [5] from $V(\mathscr{P})$ to the fiber $Z^1(\Delta_{\mathscr{P}}, \underline{\widehat{\mathbb{X}}})$ over $\mathcal{M} \in \operatorname{Pic}^\lambda A$. Consider the image of this map as a subset in M^{fr} and denote it by $\mathfrak{Z}$. It suffices to show that the intersection of $\mathfrak{Z}$ with the group orbits is discrete.

First, consider the group orbits generated by changing the projections to A. If the projection to A is changed by an element $a \in A$, then $\mathcal{M}$ is changed to $T_{-a}^*\mathcal{M}$. However, $\mathcal{M}$ is ample, there are only finitely many $a \in A$ such that $\mathcal{M} \cong T_{-a}^*\mathcal{M}$. Therefore, this group orbit intersects the fiber $Z^1(\Delta_{\mathscr{P}}, \underline{\widehat{\mathbb{X}}})$ over $\mathcal{M} \in \operatorname{Pic}^\lambda A$ at finitely many points, thus intersects $\mathfrak{Z}$ at only finitely many points.

Secondly, the action of $C^0(\Delta_{\mathscr{P}}, \underline{\widehat{\mathbb{X}}})$ on the fiber $Z^1(\Delta_{\mathscr{P}}, \underline{\widehat{\mathbb{X}}})$ over $\mathcal{M} \in \operatorname{Pic}^\lambda A$. For any $t \in C^0(\Delta_{\mathscr{P}}, \underline{\hat{\mathbb{X}}})$, it maps ψ_0 to ψ_0/t[6] and leaves τ_0 unchanged. However, there is no positive dimensional locus where τ_0 is constant in $V(\mathscr{P})$. Therefore, the intersection is also discrete.

In sum, there is no positive dimensional locus in $V(\mathscr{P})$ that is mapped

[5]Here we mean every fiber of the map is finite.

[6]Use the notations in (loc. cit. Definition 5.3.4), ψ_0 is a function over $C_0(\Delta_{\mathscr{P}}, \underline{\mathbb{X}})$.

to a point in the groupoid of polarized SSAV, and our claim is proved. $\square$

Remark 4.66. Although the algebraic stack $\overline{\mathscr{A}}_{g,d}[1/d]$ is normal over $\mathbf{Z}[1/d]$, we are unable to prove that its coarse moduli space is also normal. The problem is we don't know if $\overline{\mathscr{A}}_{g,d}$ is Deligne–Mumford or not. Therefore, locally, we can only choose a finite flat groupoid representation $R \to U \times U$, instead of an étale representation. Then we don't know how to choose U to be normal.

Chapter 5

The Degeneration as Stable Pairs

Fix the base field $k = \mathbf{C}$ in this section.

5.1 The Heisenberg Relation and the Fourier Decomposition

Let's study a single abelian variety first. Fix a polarized abelian variety (G, λ) over $\mathbf{C}$. Suppose the polarization $\lambda : G \to G^t$ is induced by an ample line bundle $\mathcal{L}$. Let $K(\lambda)$ be the kernel of λ. The automorphism group of the pair $(G, \mathcal{L})$ over the translations of G is the theta group $\mathcal{G}(\mathcal{L})$. $\mathcal{G}(\mathcal{L})$ is a central extension of $K(\lambda)$ by $\mathbf{C}^*$

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{G}(\mathcal{L}) \xrightarrow{p} K(\lambda) \longrightarrow 0.$$

The space of global sections $\Gamma(G, \mathcal{L})$ is an irreducible $\mathcal{G}(\mathcal{L})$-module. We refer the reader to [Mum67] for the basic facts about the theta group.

Write G as V/Λ, where V is the universal cover of G and $\Lambda = \mathrm{H}_1(G, \mathbf{Z})$. The polarization is represented by an integral skew symmetric form E over Λ. Define $\Lambda^\vee := \{v \in V; E(v, v') \in \mathbf{Z}, \forall v' \in \Lambda\}$.

Definition 5.1. Define $\mathcal{H}(V,E)$ to be a central extension of V by $\mathbf{C}^*$,

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{H}(V,E) \xrightarrow{p} V \longrightarrow 0.$$

that induces the pairing $\exp(-2\pi i E)$ on V. In other words, it is the Heisenberg group with the commutator

$$(t,v)(t',v')(t,v)^{-1}(t',v')^{-1} = \big(\exp(-2\pi i E(v,v')),0\big), \tag{5.1}$$

Fix a Lagrangian $U \subset V$. Denote $U \cap \Lambda$ by X^* and Λ/X^* by Y. Choose a classical factor of automorphy $e_{\mathcal{L}} \in Z^1(\Lambda, \Gamma(V, \mathcal{O}_V^*))$ that is trivial over X^* (See [BL04] p. 50 for notations.)

$$e_{\mathcal{L}}(\lambda,v) = \chi(\lambda)\exp\left(\pi(H-B)(v,\lambda) + \frac{\pi}{2}(H-B)(\lambda,\lambda)\right), \quad \lambda \in \Lambda, v \in V.$$

The semi-character χ defines a lift $\Lambda \to \mathcal{H}(V,E)$ via $\lambda \mapsto (\chi^{-1}(\lambda),\lambda)$. From now on, we regard Λ as a subgroup of $\mathcal{H}(V,E)$. Notice that X^* is lifted to $(1,X^*)$. Take the normalizer $N(\Lambda)$ of Λ and the normalizer $N(X^*)$ of X^* in $\mathcal{H}(V,E)$.

Consider $h : \Lambda \times V \to \mathbf{C}^*$ defined by

$$h(\lambda,v) := \exp\left(\pi(H-B)(v,\lambda) + \frac{\pi}{2}(H-B)(\lambda,\lambda)\right), \quad \lambda \in \Lambda, v \in V.$$

Extend h to $V \times V \to \mathbf{C}^*$. Then h satisfies

$$h(v_1+v_2,v')e^{\pi i E(v_1,v_2)} = h(v_1,v'+v_2)h(v_2,v'). \tag{5.2}$$

Define the action of $\mathcal{H}(V, E)$ on the trivial bundle $\mathbf{C} \times V$

$$S_{(t,v)}(t', v') = (t^{-1}t'h(v, v'), v + v'). \tag{5.3}$$

This is a group representation of $\mathcal{H}(V, E)$ by Relation (5.2). Since Λ (resp. X^*) is in the center of $N(\Lambda)$ (resp. $N(X^*)$), the action S induces an action of $N(\Lambda)/\Lambda$ (resp. $N(X^*)/X^*$) on the quotient $(G, \mathcal{L}) = (V, \mathbf{C} \times V)/\Lambda$ (resp. $(T, \mathbf{C} \times T) = (V, \mathbf{C} \times V)/X^*$). Notice that if $\mathcal{L}$ is considered as the coherent sheaf of sections instead of the line bundle, then the action of the center $\mathbf{C}^* \subset N(\Lambda)/\Lambda$ is of weight 1. Therefore, the action of $N(\Lambda)/\Lambda$ on $\Gamma(G, \mathcal{L})$ is identified with the action of $\mathcal{G}(\mathcal{L})$ on $\Gamma(G, \mathcal{L})$. On the other hand, $N(X^*)/X^*$ contains the subgroup $(1, U/X^*)$. By ([BL04] Lemma 3.2.2. a)), $h(U, V) = 1$. It follows that the action (5.3) of $a \in (1, U/X^*)$ on $\mathbf{C} \times T$ is $T_a^* : (t, v) \mapsto (t, av)$. Therefore we also identify the action of $(1, U/X^*)$ on $\mathbf{C} \times T$ with the action of the real subtorus $U/X^* \subset V/X^* = T$ on $\mathbf{C} \times T$. Define $\phi : \Lambda^\vee \to X$ by $\phi(\lambda)(u) = E(\lambda, u)$ for $\lambda \in \Lambda^\vee$, $u \in X^*$. We generalize the Heisenberg relation in [Mum72].

Lemma 5.2 (Heisenberg relation I)**.** *Consider the two groups $N(\Lambda)/X^*$ and T acting on the trivial line bundle $\mathbf{C} \times T$. Denote the action of $N(\Lambda)/X^*$ by S, and the action of T by T, we have*

$$T_a^* S_g^* = \mathrm{X}^{\phi(p(g))}(a) \cdot S_g^* T_a^*, \quad \forall g \in N(\Lambda)/X^*, a \in T, \tag{5.4}$$

where p is the projection $N(\Lambda)/X^* \to \Lambda^\vee/X^*$.

Proof. It suffices to check (5.4) for the real points $a \in U/X^* \subset T$. In this case, we can identify it as an element in $N(X^*)/X^*$, and use the commutation relation (5.1) in the Heisenberg group $\mathcal{H}(V, E)$ for $N(\Lambda)$ and $(1, U)$. $\square$

We can generalize this relation to a general cusp F_ξ. Let U_ξ be a rational isotropic subspace of dimension r. Represent G as $\widetilde{G}/Y_\xi$, where $\widetilde{G} = V/\Lambda \cap U_\xi^\perp$ and $Y_\xi = \Lambda/\Lambda \cap U_\xi^\perp$. Denote $\Lambda \cap U_\xi$ by X_ξ^* and its dual by X_ξ. E induces the polarization $\phi : Y_\xi \to X_\xi$. The algebraic torus $T_\xi = Hom(X_\xi, \mathbf{C}^*)$ is the torus part of the semi-abelian variety $\widetilde{G}$.

Choose a Lagrangian $U \supset U_\xi$. We have inclusions $X_\xi^* \subset X^*$, $T_\xi \subset T$, and quotients $X \to X_\xi$, $Y \to Y_\xi$. The free group $Y' := U_\xi^\perp \cap \Lambda/X^*$ is the kernel of $Y \to Y_\xi$. Then $\Lambda^\vee/U_\xi^\perp \cap \Lambda$ (resp. $(\widetilde{G}, \widetilde{\mathcal{L}})$) is the quotient of $\Lambda^\vee/X^*$ (resp. $(T, T \times \mathbf{C})$) by Y'. Y' is naturally considered as a subgroup of $N(\Lambda)/X^*$ by the lift χ^{-1}. Then $N(\Lambda)/U_\xi^\perp \cap \Lambda$ is a quotient of $N(\Lambda)/X^*$ by Y'. The following lemma is just the Y'-quotient of Lemma 5.2. Notice that by the relation (5.4), Y' commutes with T_ξ on $T \times \mathbf{C}$. Therefore the action of T_ξ, and the relation (5.4) descends to $(\widetilde{G}, \widetilde{\mathcal{L}})$.

Lemma 5.3 (Heisenberg relation II)**.** *Consider the two groups $N(\Lambda)/U_\xi^\perp \cap \Lambda$ and T_ξ acting on $(\widetilde{G}, \widetilde{\mathcal{L}})$. Denote the action of $N(\Lambda)/U_\xi^\perp \cap \Lambda$ by S, and the action of T_ξ by T, we have*

$$T_a^* S_g^* = \mathrm{X}^{\phi(p(g))}(a) \cdot S_g^* T_a^*, \quad \forall g \in N(\Lambda)/U_\xi^\perp \cap \Lambda, a \in T_\xi, \tag{5.5}$$

where p is the projection $N(\Lambda)/(U_\xi^\perp \cap \Lambda) \to \Lambda^\vee/U_\xi^\perp \cap \Lambda$, and ϕ is the induced map $\Lambda^\vee/U_\xi^\perp \cap \Lambda \to X_\xi$ by E.

The theta group $\mathcal{G}(\mathcal{L}) = N(\Lambda)/\Lambda$ is the quotient of $N(\Lambda)/U_\xi^\perp \cap \Lambda$ by Y_ξ. Denote the subgroup $\Lambda^\vee \cap U_\xi/\Lambda \cap U_\xi$ by K_2. The restriction of the Weil pairing to K_2 induces the surjection $K(\lambda) \to \widehat{K}_2$. Compose with $p : \mathcal{G}(\mathcal{L}) \to K(\lambda)$, we have $w : \mathcal{G}(\mathcal{L}) \to \widehat{K}_2$. Denote the kernel of w by K_w. Identify $\widehat{K}_2$ with $X_\xi/\phi(Y_\xi)$. Let I be an $X_\xi/\phi(Y_\xi)$-torsor.

Lemma 5.4. *Consider the canonical representation $\mathrm{H}^0(G, \mathcal{L})$ of $\mathcal{G}(\mathcal{L})$. $\mathrm{H}^0(G, \mathcal{L}) = \oplus_{\alpha \in I} V_\alpha$ is decomposed into irreducible representations of K_w, labelled by I. This is the same decomposition as the Fourier decomposition*

$$\mathrm{H}^0(G, \mathcal{L}) = \bigoplus_{\alpha \in X_\xi/\phi(Y_\xi)} \mathrm{H}^0(A, \mathcal{M}_\alpha)$$

in the AN construction. Moreover, for $g \in \mathcal{G}(\mathcal{L})$, the action of g translates the space labelled by α to the space labelled by $\alpha + w(g)$.

Proof. Recall that the Fourier decomposition is obtained by the Fourier decomposition of $\mathrm{H}^0(\widetilde{G}, \widetilde{\mathcal{L}})$ from the action of T_ξ. It follows from Lemma 5.3 that the action of K_w preserves this decomposition. By ([Ols08] Theorem 5.4.2.), the theta group $\mathcal{G}(\mathcal{M})$ is contained in the subgroup K_w. Since each $\mathrm{H}^0(A, \mathcal{M}_\alpha)$ is already an irreducible representation of the subgroup $\mathcal{G}(\mathcal{M})$, it is irreducible for K_w. Since $w = \phi \circ p$, the last sentence follows from Equation (5.5). $\square$

5.2 The Stable Pairs

It is not hard to generalize the above picture to the relative case. Let $\pi : \mathcal{X} \to S$ be a torsor for an abelian scheme G over a locally Noetherian base S, with a relatively ample invertible sheaf $\mathcal{L}$ over $\mathcal{X}$. Let $K(\lambda)$ be the kernel of the polarization $\lambda : G \to G^t$ given by $\mathcal{L}$. $K(\lambda)$ is a finite flat group scheme over S. Define $\mathcal{G}(\mathcal{L})$ to be the functor whose T-points over $T \to S$ are the group of automorphisms of $(\mathcal{X}_T, \mathcal{L}_T)/T$ that commute with G_T-actions. The functor $\mathcal{G}(\mathcal{L})$ is represented by a group scheme called the theta group for $\mathcal{L}$, and is defined in ([Mum67] Section 6). It can also be realized as follows. Choose a local section of $\mathcal{X}$ and regard the line bundle $\mathcal{L}$ as a $\mathbb{G}_m$-torsor over G. Restricted to $K(\lambda)$, $\mathcal{L}$ is a central extension by $\mathbb{G}_m$ and is the group $\mathcal{G}(\mathcal{L})$ ([Mum67] Section 6 Propostion 1).

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(\mathcal{L}) \xrightarrow{p} K(\lambda) \longrightarrow 0. \tag{5.6}$$

The commutators in $\mathcal{G}(\mathcal{L})$ give the Weil pairing $e^{\mathcal{L}}$ on $K(\lambda)$. By ([Shi12] Proposition 2.12, 2.13), $\pi_*\mathcal{L}$ is locally free of rank d and is an irreducible representation of $\mathcal{G}(\mathcal{L})$.

Lemma 5.5. *Assume over a base $S/\mathbf{C}$, both the K_w-irreducible representation decomposition and the Fourier decomposition exist for $\pi_*\mathcal{L}$, and are defined by the same cusp. Then the two decompositions are the same.*

Proof. First, when S has a $\mathbf{C}$-point, it follows from Lemma 5.4 and Lemma 5.6 below. In general, we have $S \to \mathscr{A}_{g,\delta}$. We can take the closure of the image, and find a $\mathbf{C}$-point. Prove the statements for the closed points of S, and then use Lemma 5.6. $\square$

Lemma 5.6. *Let S be a Noetherian scheme, and $\mathcal{E}$ be a locally free coherent sheaf over S. Assume there are two decompositions $\mathcal{E} = \oplus\mathcal{M}_\alpha = \oplus\mathcal{N}_\alpha$ indexed by the same set I, such that $\mathcal{M}_\alpha \otimes \kappa(x) = \mathcal{N}_\alpha \otimes \kappa(x)$ for each closed point $x \in S$. Then the two decompositions of $\mathcal{E}$ agree over S.*

Proof. Since the problem is local, assume $S = \operatorname{Spec} R$, for R with the residue field κ, and $\mathcal{M}_\alpha \otimes \kappa = \mathcal{N}_\alpha \otimes \kappa$. Then $\mathcal{E} = \widetilde{E}$ (resp. $\mathcal{M}_\alpha$, $\mathcal{N}_\beta$) for a finitely generated R-module E (resp. M_α, N_β). Since R is local, all E, M_α, N_β are free. Moreover, we have the morphisms $p_{\alpha\beta} : M_\alpha \to N_\beta$ and $q_{\beta\alpha} : N_\beta \to M_\alpha$ that are defined as the compositions of the inclusions and the projections. Consider the image $\operatorname{Im}(p_{\alpha\beta})$ of the morphism $p_{\alpha\beta} : M_\alpha \to N_\beta$ when $\alpha \neq \beta$. Since both M_α and N_β are free, the tensor $\operatorname{Im}(p_{\alpha\beta}) \otimes \kappa$ is the image of the morphism $p_{\alpha\beta} \otimes \kappa : M_\alpha \otimes \kappa \to N_\beta \otimes \kappa$, which is zero. By Nakayama's lemma, $\operatorname{Im}(p_{\alpha\beta})$ is zero, and $p_{\alpha\beta} = 0$. It follows that M_α is included in N_α. Similarly N_α is included in M_α, and we have $M_\alpha = N_\alpha$ inside E. $\square$

Let $M = 2\delta_g$. Consider the map

$$P_M : \mathcal{G}(\mathcal{L}) \to \mathcal{G}(\mathcal{L}) \tag{5.7}$$

$$g \mapsto g^M \quad \forall g \in \mathcal{G}(\mathcal{L})(S). \tag{5.8}$$

Lemma 5.7. *The map P_M is a group homomorphism.*

Proof. Fix a scheme S, and consider $g, h \in \mathcal{G}(\mathcal{L})(S)$. Recall the commutator $ghg^{-1}h^{-1} = e^{\mathcal{L}}(p(g), p(h))$. By induction, we have

$$g^n h^n = (e^{\mathcal{L}}(p(g), p(h)))^{n-1} ghg^{n-1}h^{n-1} = (e^{\mathcal{L}}(p(g), p(h)))^{n(n-1)/2}(gh)^n.$$

Since the order of $e^{\mathcal{L}}(p(g), p(h))$ divides δ_g, $P_M(g)P_M(h) = P_M(gh)$. $\square$

Let $G(M)$ denote the kernel of P_M. It is a subgroup scheme of $\mathcal{G}(\mathcal{L})$. Since μ_M is defined as the kernel of the group homomorphism

$$\mathbb{G}_m \to \mathbb{G}_m \tag{5.9}$$

$$g \mapsto g^M \quad \forall g \in \mathbb{G}_m(S), \tag{5.10}$$

we have

$$1 \longrightarrow \mu_M \longrightarrow G(M) \xrightarrow{p} K(\lambda) \longrightarrow 0. \tag{5.11}$$

Let $\mathcal{H}(\delta, M)$ be the Heisenberg group. It is a central extension of $H(\delta) \times \widehat{H(\delta)}$ by μ_M. The multiplication is determined by the requirement that if $\tilde{g}, \tilde{h}$ are lifts of $g \in H(\delta)$ and $h \in \widehat{H(\delta)}$ in $\mathcal{H}(\delta, M)$, then the commutator $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1} = h(g)$. Étale locally, $G(M)$ is isomorphic to the constant group $\mathcal{H}(\delta, M)$.

Assumption 5.8. *Suppose $S = \operatorname{Spec} R$ for R Noetherian, normal integral, complete, local, $\mathbf{C}$-algebra, with generic point $\eta = S^*$, closed point S_0. Denote the field of fractions by K.*

Assume $\pi : (\mathcal{X}, G, \mathcal{L}, \varrho) \to S$ a polarized stable semiabelic scheme, such that the generic fiber $\pi : (\mathcal{X}^*, \mathcal{L}^*) \to S^*$ is a polarized abelian variety. By [FC90] Chap. II Sect. 2, we have two Raynaud extensions $0 \to T \to \widetilde{G} \to A \to 0$ and $0 \to T^t \to \widetilde{G}^t \to A^t \to 0$ from the polarized semi-abelian scheme G. Let X_ξ (resp. Y_ξ) be the character group of T (resp. T^t). The restriction of the polarization to the toric part is $\lambda_T : T \to T^t$ and is induced by $\phi : Y_\xi \to X_\xi$. Let the kernel of λ_T be K_2. It is an isotropic finite subgroup scheme of the kernel $K(\lambda)$ and is denoted by $K_S(\mathcal{L}_\eta)^m$ in [Nak99]. The Cartier dual is $\widehat{K}_2 = X_\xi/\phi(Y_\xi)$ and is well-defined for the family. The restriction of the Weil pairing defines a surjection $K(\lambda) \to \widehat{K}_2$. Composing with $G(M) \to K(\lambda)$, we get a homomorphism $w : G(M) \to \widehat{K}_2$. Let K_w be the kernel of w. Consider the irreducible $G(M)$-representation $\pi_*\mathcal{L}^*$. After a base change, étale over S^*, $G(M)$ and $\widehat{K}_2$ are constant, $\pi_*\mathcal{L}^*$ decomposes into irreducible K_w-representations $\pi_*\mathcal{L}^* = \oplus_{\alpha\in I}\mathcal{V}_\alpha$, where I is a torsor for the group $\widehat{K}_2$. For any $g \in G(M)$, the action of g translates $\mathcal{V}_\alpha \to \mathcal{V}_{\alpha+w(g)}$. Fix an element of I and identify I with the group $\widehat{K}_2$. A lift $\sigma : \widehat{K}_2 \to G(M)$ is a section for the map $w : G(M) \to \widehat{K}_2$. We do not require σ to be a group homomorphism.

Proposition 5.9. *If S satisfies Assumption 5.8 and $\pi : (\mathcal{X}, G, \mathcal{L}, \varrho)/S$ is the pull-back of the AN family along a morphism $g : S \to \overline{\mathscr{A}}^m_{g,\delta}$, then after an étale base change, we can extend $G(M)$ and K_w-representation $\pi_*\mathcal{L} = \oplus_{\alpha\in I}\mathcal{V}_\alpha$ over S. Take any section $\vartheta_0 \in \mathcal{V}_0^*$, and any lift $\sigma : \widehat{K}_2 \to G(M)$. Define*

$$\vartheta := \sum_{\alpha\in I} S^*_{\sigma(\alpha)}\vartheta_0. \tag{5.12}$$

Let Θ be the zero locus of ϑ. Then $(\mathcal{X}, G, \mathcal{L}, \Theta, \varrho)/S$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.

Proof. Making an étale base change if necessary, we can assume $g : S \to \Xi_{\mathscr{P}}$ over the cusp F_ξ for some bounded paving $\mathscr{P}$ on U_ξ^*. The family $(\mathcal{X}, \mathcal{L}, G, \varrho)/S$ is the pull-back of AN family along g. Since AN construction is functorial (Proposition 4.19), $(\mathcal{X}, \mathcal{L}, G, \varrho)/S$ is constructed from the following data. There is an exact sequence of abelian sheaves

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^* & \longrightarrow & \widetilde{G}^* & \longrightarrow & A^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T & \longrightarrow & \widetilde{G} & \longrightarrow & A & \longrightarrow & 0 \end{array}, \tag{5.13}$$

where the top line is the restriction to S^*. Recall $X_\xi = (U_\xi \cap \Lambda)^*$, $Y_\xi = \Lambda/\Lambda \cap U_\xi^\perp$ and $\phi : Y_\xi \to X_\xi$. Moreover, we have c, c^t, τ, ψ, $\mathcal{M}$. Over S^*, $\widetilde{G}^* = \mathrm{Spec}_{A^*} \bigoplus_{\alpha \in X_\xi} \mathcal{O}_\alpha$, and the line bundle $\widetilde{\mathcal{L}}^*$ is defined by

$$\mathcal{S} := \prod_{d \geqslant 0} \left(\bigoplus_{\alpha \in X_\xi} \mathcal{O}_\alpha \otimes \mathcal{M}^d \theta^d \right) \otimes \mathcal{O}_{A^*} = \bigoplus_{(d,\alpha) \in S(X_\xi)} \mathcal{S}_{d,\alpha} \theta^d.$$

Over S, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is defined by the graded $\mathcal{O}_A$-algebra

$$\mathcal{R} := \bigoplus_{(d,\alpha,p) \in Q_{\tilde{\varphi}}} \mathrm{X}^p \otimes \mathcal{O}_\alpha \otimes \mathcal{M}^d \theta^d = \bigoplus_{(d,\alpha) \in S(X_\xi)} \mathcal{R}_{d,\alpha} \theta^d.$$

The degree-1 part is the Fourier decomposition. By Lemma 5.5, it agrees with the pull back of the decomposition $\oplus \mathcal{V}_\alpha$. In other words, $\mathcal{V}_\alpha = \pi_* \mathcal{M}_{\alpha,\eta}$ over S^*. The inclusion $\mathcal{R} \to \mathcal{S}$ is defined by the graph of the $\mathscr{P}$-piecewise affine function φ. As the pull-back of $\varphi_{\mathscr{P}}$, φ is X_ξ-quasiperiodic.

For $\lambda \in Y_\xi$, the action S^*_λ on $\widetilde{G}^*$ is expressed as

$$\psi(\lambda)^m \tau(\lambda, \alpha) : T^*_{c^t(\lambda)}(\mathcal{M}^m \otimes \mathcal{O}_\alpha) \to \mathcal{M}^m \otimes \mathcal{O}_{\alpha + m\phi(\lambda)},$$

for $\psi(\lambda) = a(\lambda)\psi'(\lambda)$, $\tau(\lambda, \alpha) = b(\lambda, \alpha)\tau'(\lambda, \alpha)$, with ψ', τ' trivializations over S and with a, b having values in $\mathcal{O}_S$. Since $\{S^*_\lambda\}$ is a group action, a, b satisfy relations

$$a(0) = 1 \tag{5.14}$$

$$a(\lambda + \mu) = b(\lambda, \phi(\mu))a(\lambda)a(\mu). \tag{5.15}$$

Consider a discrete valuation v associated with a height 1 prime ideal. Let $A_\varphi(v) : X_\xi \to \mathbf{Z}$ be the quadratic function associated with $v \circ \varphi$. Let $A(v) : Y_\xi \to \mathbf{Z}$ be the composition $v \circ a$ and $B(v) : Y_\xi \times X_\xi \to \mathbf{Z}$ be the composition $v \circ b$. Then $A(v)$ is quadratic over Y_ξ with the associated bilinear form $B(v)(\cdot, \phi(\cdot))$. We have $A(v)_\varphi|_{Y_\xi} = A(v)$.

Over $\mathbf{C}$, we can regard $G(M)$ as a subgroup of $\mathcal{H}(\delta)$[1], and present it as $N'(\Lambda)/\Lambda$ for $N'(\Lambda)$ a subgroup of $N(\Lambda)$. The following diagram is Cartesian.

$$\begin{tikzcd} N'(\Lambda)/\Lambda \cap U_\xi^\perp \arrow[r, "w"] \arrow[d] & X_\xi \arrow[d] \\ G(M) \arrow[r, "w"] & \widehat{K}_2 \end{tikzcd} . \tag{5.16}$$

The vertical arrows are quotients by Y_ξ. The group action of $G(M)$ on $(\mathcal{X}^*, \mathcal{L}^*)$ is represented by the group action of $N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$ on $(\widetilde{G}^*, \widetilde{L}^*)$

[1]More precisely, all the groups appearing here should be the constant groups times S^*, for simplicity, we ignore the notation $\times S^*$.

which extends the action of Y_ξ on $(\widetilde{G}^*, \widetilde{L}^*)$. Write the action of $N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$ also in the form $a^d b(\psi')^d \tau'$,[2] and take the valuations $A_X(v) := v \circ a$ and $B_X(v) := v \circ b$. These two functions factor through X_ξ and $X_\xi \times X_\xi$. This is because for different lifts of elements from X_ξ to $N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$, the differences are in μ_M, and they have the same order under v. Since a, b are defined from group actions, they satisfy the same relations as Relation (5.14). Therefore $A_X(v)$ is also a quadratic function over X_ξ extending $A(v)$, and $A_\varphi(v) = A_X(v)$ for all v.

Since R is a normal Noetherian domain, $R = \cap_{\mathrm{ht}\,\mathfrak{p}=1} R_\mathfrak{p}$ ([Mat86] Theorem 11.5). It follows that the difference between the quadratic part of $\mathrm{X}^{\varphi(w(g))}$ and $a(g)$ is invertible in R for all $g \in N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$. Therefore the values of a, b for $N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$ are regular functions over S. The action of $G(M)$ is defined on $(\mathcal{X}, \mathcal{L})$ over S and S_g^* maps $\mathcal{R}_{0,1}$ to $\mathcal{R}_{\phi(g),1}$ for any $g \in N'(\Lambda)/(U_\xi^\perp \cap \Lambda)$. In particular, $S_g^*(\vartheta_0)$ is in $(\pi_* \mathcal{M}_{\phi(g)})^*$ over S. Therefore, for any lift $\sigma : I \to G(M)$, the section

$$\vartheta := \sum_{\alpha \in I} S^*_{\sigma(\alpha)} \vartheta_0$$

is stable. The family $(\mathcal{X}, G, \Theta, \varrho)/S$ is in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$ by Theorem 4.25. $\square$

Definition 5.10. If an isotropic subgroup K_2 of $K(\lambda)$ or $\widehat{K}_2 = X_\xi/\phi(Y_\xi)$ is well-defined for the family, call the set of divisors obtained in Proposition 5.9 the balanced set and denote it by $S(K_2)$.

[2] In other words, ψ', τ' are trivializations over S.

Remark 5.11. The isotropic subgroup K_2 is well-defined for the family when the base is very local, for example, when S satisfies Assumption 5.8. It is also well-defined when the interior S^* is a punctured polydisc $(\Delta^*)^n$ ([CCK79] Proposition 2.1) $K_2 = \Lambda^\vee \cap W_0/\Lambda \cap W_0$.

Corollary 5.12. If the base S is a DVR, and we have a polarized abelian variety $(G^*, \mathcal{X}^*, \mathcal{L}^*)$ over the generic point, we can add the central fiber as follows. The monodromy defines an isotropic subgroup K_2. Pick any divisor Θ^* from the balanced set $S(K_2)$, we get an object $(G^*, \mathcal{X}^*, \Theta^*)$ in $\overline{\mathscr{A}\mathscr{P}}_{d,g}$. Since $\overline{\mathscr{A}\mathscr{P}}_{d,g}$ is proper, we can uniquely extend the family $(G, \mathcal{X}, \Theta)$ over S. Then forget about the divisor Θ, we get a family $(\mathcal{X}, G, \mathcal{L}, \varrho)$ over S. This is the pull-back family from $\overline{\mathscr{A}}^m_{g,\delta}$, and is independent of the choice of $\Theta \in S(K_2)$.

Proof. Since $\overline{\mathscr{A}}^m_{g,\delta}$ is also proper, we have a unique morphism $S \to \overline{\mathscr{A}}^m_{g,\delta}$ and we can pull back the AN family. By Proposition 5.9, the pull-back family coincides with $(\mathcal{X}, G, \mathcal{L}, \varrho)$. □

The following is the geometric description of our extended families near a cusp F_ξ.

Theorem 5.13. *Suppose S satisfies Assumption 5.8 and in addition R strictly henselian. Let $(\mathcal{X}, G, \mathcal{L}, \varrho)/S$ be a polarized stable semiabelic scheme over S, with the generic fiber abelian. Over the generic point η, the group subscheme $K_2 \subset K(\lambda)_\eta$ and the stable set of divisors $S(K_2)$ are defined by the Raynaud* *extension. Then $(\mathcal{X}, G, \mathcal{L}, \varrho)$ is the pull-back of the AN family along a unique*[3] *morphism $g : S \to \overline{\mathscr{A}}^m_{g,\delta}$ if and only if the group scheme $G(M)$ can be extended over S (thus $S(K_2)$ is defined over S), and for one (equivalently any) divisor Θ from $S(K_2)$, $(\mathcal{X}, G, \Theta, \varrho)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.*

Proof. The "only if" part is Proposition 5.9. Suppose there exists $\Theta^* \in S(K_2)$ such that $(\mathcal{X}, G, \Theta, \varrho)$ is an object in $\overline{\mathscr{A}\mathscr{P}}_{g,d}$. This is equivalent to a morphism $f : S \to \overline{\mathscr{A}\mathscr{P}}_{d,g}$. The semi-abelian scheme G/S gives rise to étale constructible sheaves $\underline{X}$, $\underline{Y}$, polarization $\phi : \underline{Y} \to \underline{X}$ and pairing $\underline{B} : \underline{Y} \times \underline{X} \to \underline{\mathrm{Div}} S$. Over the closed point S_0, the fiber is X_ξ, Y_ξ, $\phi : Y_\xi \to X_\xi$ and $b : Y_\xi \times X_\xi \to K^*$. Over any $s \in S$, $\underline{X}_s$ (resp. $\underline{Y}_s$) is the quotient of X_ξ (resp. Y_ξ).

For each point $s \in S$, choose a DVR $T \to S$ with the generic point mapped to η and the closed point mapped to s. By Corollary 5.12, the pull-back family over T, denoted by $(\mathcal{X}_T, G_T, \Theta_T, \varrho_T)$, comes from AN family over $\overline{\mathscr{A}}^m_{g,\delta}$. In particular, $(\mathcal{X}_T, G_T, \Theta_T, \varrho_T)$ is constructed from an $\underline{X}_s$-quasiperiodic piecewise affine function $\varphi_T : \underline{X}_{s,\mathbf{R}} \to \mathbf{R}$. Therefore the paving $\mathscr{P}_s$ associated with φ_T is $\underline{X}_s$-invariant. Let b_s be $\underline{B}_s$ and v the discrete valuation of T. The bilinear form $v \circ b_s$ is inside the cone $C(\mathscr{P}_s)$ of the second Voronoi fan $\Sigma(\underline{X}_s)$. The paving $\mathscr{P}_s$, regarded as a cell decomposition of $\underline{X}_{s,\mathbf{R}}/\phi(\underline{Y}_s)$, is the cell complex Δ_0 in ([Ale02] Definition 5.7.2) and can be defined by the fiber $(\mathcal{X}_s, \mathcal{L}_s)$. Moreover, since s specializes to S_0, if we pull back the paving $\mathscr{P}_s$

[3]For a morphism to a stack, "being unique" means that, if there is another morphism g' with the same properties, the morphism (g, g') factors through the diagonal.

along $X_\xi \to \underline{X}_s$, it is coarser than the paving $\mathscr{P}$ associated with the central fiber $(\mathcal{X}_0, \mathcal{L}_0)$. Therefore the cone $C(\mathscr{P}_s)$ is a face of $C(\mathscr{P})$ in $\Sigma(X_\xi)$. Write X_ξ as a quotient of $X = \mathbf{Z}^g$. In particular, if $v : K^* \to \mathbf{Z}$ is any discrete valuation defined by a prime ideal of height one, then $v \circ b$ is contained in the closed cone $C(\mathscr{P})$ in the second Voronoi fan $\Sigma(X)$. By Proposition 4.60, there exists a morphism $g : S \to \overline{\mathscr{A}}^m_{g,\delta}$ such that if $(\mathcal{X}', \mathcal{L}', G', \varrho')$ is the pull-back of the AN family along g, then $G' \cong G$. Take the closure Θ' of Θ^* for $(G', \mathcal{X}', \mathcal{L}', \varrho')$, so that $\Theta'|_{S^*}$ is identified with $\Theta|_{S^*}$ through the isomorphism. By Proposition 5.9, we get another morphism $g' : S \to \overline{\mathscr{A}\mathscr{P}}_{g,d}$. Consider the cartesian diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "h"'] & \overline{\mathscr{A}\mathscr{P}}_{g,d} \arrow[d, "\Delta"] \\ S \arrow[r, "(f,g')"] & \overline{\mathscr{A}\mathscr{P}}_{g,d} \times \overline{\mathscr{A}\mathscr{P}}_{g,d} \end{tikzcd}$$

Since Δ is finite ([Ale02] Theorem 5.10.1), h is finite. Moreover, $S^* \to S$ factors through h. Since the local charts for $\overline{\mathscr{A}\mathscr{P}}_{g,d}$ are integral ([Ale02] 5.9, they are semigroup k-algebras), S' is integral. By Lemma 5.14, $S' = S$. The families $(\mathcal{X}, \mathcal{L}, G, \Theta, \varrho)$ and $(\mathcal{X}', \mathcal{L}', G', \Theta', \varrho')$ are isomorphic.

The morphism $g : S \to \overline{\mathscr{A}}^m_{g,\delta}$ thus obtained is unique, because the diagonal of $\overline{\mathscr{A}}^m_{g,\delta}$ is also finite. We use the same argument as above. $\square$

Lemma 5.14. *Let $h_1 : R \to R'$ be a finite extension of integral domains and $h_2 : R' \to K$ be a morphism such that the composition $h_2 \circ h_1 : R \to K$ is the inclusion of R into its field of fractions. If R is normal, then h_1 is an isomorphism.*

Proof. We claim that h_2 is injective. Suppose there is $0 \neq b \in R'$ that is mapped to zero in K. Since b is integral over R, we have $b^n + a_1 b^{n-1} + \ldots + a_n = 0$ for $a_i \in R$ and $a_n \neq 0$. However, this implies that a_n is mapped to 0 in K, a contradiction. Therefore $R \subset R' \subset K$. If R is normal, then $R = R'$. $\square$

Remark 5.15. Heuristically, locally near the boundary, the compactification $\overline{\mathscr{A}}^m_{g,\delta}$ should be the normalization of a slice of $\overline{\mathscr{A}\mathscr{P}}_{g,d}$. The slice is defined by a choice of a divisor in the balanced set $S(K_2)$. The divisors in $S(K_2)$ should be regarded to be the most "symmetric" because it is an average over the lift $\sigma(\widehat{K}_2)$. The tropical avatar of the slice is the linear section σ in Equation (3.17). If we forget about the divisor Θ and only consider the family $(\mathcal{X}, \mathcal{L}, G, \varrho)$, then the normalization doesn't depend on the choice of Θ.

Remark 5.16. The families have been studied in [Nak10] and [Ols08]. Moreover, when the polarization is separable, it is proved in ([Nak99] Definition 5.11, Lemma 5.12) and [Ols08] that $G(M)$ and its representation can be extended to the boundary. What we are suggesting here is, one can characterize the extended families by using $G(M)$ and $\overline{\mathscr{A}\mathscr{P}}_{g,d}$.

Remark 5.17. However, we can not get a moduli functor this way, because the balanced set $S(K_2)$ depends on K_2. Even when the coarse moduli space has only one cusp, K_2 is never well–defined over the whole moduli space unless the polarization is principal. To parallel transport W_0, we need the Gauss–Manin connection. That means, we need an actual family over the moduli space. In all cases, we will need at least a finite ramified cover to get the family and there will be more than one cusp. Then it can be proved that there is no section stable for all K_2 unless the polarization is principal.

If K_2 is a Lagrangian, A and $\mathcal{M}$ are trivial and $\mathcal{V}_0 \cong \mathcal{O}_S$. $S(K_2)$ is a finite set. If the Lagrangian K_2 further splits, i.e., it admits an isotropic complement $K(\lambda) = K_1 \oplus K_2$, we can identify K_1 and I, and require that the lifts are group homomorphisms $\sigma : K_1 \to G(M)$. In this case, the lift $\sigma(K_1)$ is a maximal level subgroup $\widetilde{K}_1$ of $\mathcal{G}(\mathcal{L})$. There are altogether d choices, and each choice is equivalent to a choice of the descent data $h : \mathcal{X} \to \mathcal{X}/K_1$, with $\nu : h^*\mathcal{L}' \cong \mathcal{L}$ for a principally polarized line bundle $\mathcal{L}'$ over $\mathcal{X}/K_1$. The stable section ϑ is the pull-back of the unique (up to a scaling) section of $\mathcal{L}'$. The sections $S^*_{\sigma(\alpha)}\vartheta_0$ are the classical theta functions parameterized by K_1.

Appendix

Appendix 1

The Quasiperiodic Functions

Let $Y \cong \mathbf{Z}^g$ be a finitely generated free abelian group acting on an affine space $V \cong \mathbf{R}^g$ by translations. Let ψ be a real valued piecewise affine function on V.

Definition 1.1. A piecewise affine function ψ is called *quasi-periodic* with respect to Y if

$$\psi(x+\lambda) - \psi(x) = A_\lambda(x),\ \forall \lambda \in Y,\ \forall x \in V, \tag{1.1}$$

for $A_\lambda(x)$ an affine function on V that depends on $\lambda \in Y$.

Remark 1.2. A Y-quasiperiodic function ψ can be regarded as an element in $\Gamma(V/Y, \mathcal{P}A/\mathcal{A}ff)$.

Lemma 1.3. *If ψ is quasi-periodic with respect to Y, then there exists some quadratic function A such that $\psi - A$ is a Y-periodic function on V.*

Proof. Fix a point $x_0 \in V$ and regard V as a vector space. Embed Y as a subset of V. For $\lambda, \mu \in Y$, we have

$$\psi(x+\mu+\lambda) = \psi(x+\mu) + A_\lambda(x+\mu) \tag{1.2}$$

$$= \psi(x) + A_\mu(x) + A_\lambda(x+\mu) \tag{1.3}$$

$$= \psi(x) + A_\lambda(x) + A_\mu(x+\lambda) \tag{1.4}$$

$$= \psi(x) + A_{\mu+\lambda}(x). \tag{1.5}$$

From (1.4) and (1.5)

$$A_{\lambda+\mu}(x) = A_\mu(x+\lambda) + A_\lambda(x)$$

Therefore $\{A_\lambda\}$ is a 1-cocycle for the Y-module $Aff(X_{\mathbf{R}}, \mathbf{R})$.

Suppose $A_\lambda(x) = B(\lambda, x) + A(\lambda)$, for a linear function $B(\lambda, \cdot)$ on V and a constant $A(\lambda)$. From (1.3) and (1.4), we have

$$B(\mu, \lambda) = A_\mu(x+\lambda) - A_\mu(x) = A_\lambda(x+\mu) - A_\lambda(x) = B(\lambda, \mu).$$

Applying (1.5),

$$A_{\mu+\lambda}(x) - A_\mu(x) - A_\lambda(x) = B(\mu, \lambda),$$

It follows that

$$B(\mu+\lambda, x) = B(\mu, x) + B(\lambda, x), \tag{1.6}$$

$$A(\mu+\lambda) - A(\mu) - A(\lambda) = B(\mu, \lambda). \tag{1.7}$$

Therefore $B(\mu, \lambda)$ is a symmetric bilinear form on Y, and

$$A(\gamma) = \frac{1}{2}B(\gamma, \gamma) + \frac{1}{2}L(\gamma)$$

is a quadratic function associated with B. Here L is a linear function on Y.

Extend A and B to functions on V. Since $A(x+y) - A(x) = B(y, x) + A(y)$, for all $x, y \in V$,

$$\psi(x + \lambda) - A(x + \lambda) = \psi(x) - A(x), \ \forall \lambda \in Y, x \in V.$$

In other words, $\psi - A$ is a periodic function with respect to Y. □

If we forget about x_0 initially chosen, the quadratic part of A is well-defined.

Definition 1.4. The quadratic form $1/2B(x, x)$ is called the quadratic form associated with ψ.

If the piecewise affine function φ takes values in a vector space $P^{gp}_{\mathbf{R}}$, we can define Y-quasiperiodic similarly. Recall a quadratic function is an element in $\Gamma^2 V^* \otimes P^{gp}_{\mathbf{R}}$. We can generalize Lemma 1.3.

Corollary 1.5. If ψ is quasi-periodic with respect to Y, then there exists some quadratic function A such that $\psi - A$ is a Y-periodic function on V.

Appendix 2

Degenerations of One-parameter Families

2.1 Maximal Degeneration

In this section, we compute the degeneration data for a general 0-cusp. Assume the base field $k = \mathbf{C}$.

Let $\pi : \mathcal{X}^* \to \Delta^*$ be a one-parameter family of polarized abelian varieties and X be a generic fiber. Let $V := \mathrm{H}_1(X, \mathbf{R})$ and $\Lambda := \mathrm{H}_1(X, \mathbf{Z})$. The polarization is defined by a skew-symmetric integral bilinear form E over Λ. The log monodromy defines a weight filtration $0 \subset W_0 \subset W_1 \subset W_2 = V$. In our usual notations, $U = W_0$, $U^\perp = W_1$. Define $X^* := \Lambda \cap U$ and $Y := \Lambda / U^\perp \cap \Lambda$. Fix a symplectic basis $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ of Λ for E. We use (x, y) to denote the coordinates of a vector $v = x\lambda + y\mu$, where x and y are both row vectors of g elements. Assume the degeneration is maximal and U is a rational Lagrangian of dimension g. Suppose $v_1, \dots, v_g$ is a basis of X^* and

$$v_i = \sum_{j=1}^{g} c_{ij}\lambda_j + d_{ij}\mu_j, \quad \forall i = 1, \dots, g.$$

Since $\{v_1, \dots, v_g\}$ is a complex basis for V, $c\tau + d\delta$ is invertible, where c and d are $g \times g$-matrices. With respect to the complex basis $\{v_1, \dots, v_g\}$, the vector $x\lambda + y\mu$ has coordinates $(x\tau + y\delta)(c\tau + d\delta)^{-1}$.

For the convenience of computation, we change to a new basis. Suppose the dual basis of $\{v_1, \dots, v_g\}$ of X^* is a compatible basis of X. Extend $\{v_1 \dots, v_g\}$ to a basis $\{u'_1, \dots, u'_g, v_1 \dots, v_g\}$ of Λ, and under this new basis,

$$E = \begin{pmatrix} S & \mathfrak{d} \\ -\mathfrak{d} & 0 \end{pmatrix},$$

where S is an integral, skew symmetric matrix. Let the transformation matrix be $M' \in \mathrm{GL}(2g, \mathbf{Z})$. As in Corollary 3.32, we can always choose M' such that

$$\begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} M'^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(E, \mathbf{Q}),$$
$$A^{-1} = \mathfrak{d}\delta^{-1},$$
$$A^{-1}B\mathfrak{d} = \frac{1}{2}S.$$

Let (x', y') be the coordinates under the basis $\{u'_1 \dots, u'_g, v_1, \dots, v_g\}$, and (x, y) be the coordinates under the basis $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$. The transformation of coordinates is

$$\begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} M'.$$

Therefore,

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}.$$

Fix a punctured holomorphic disk Δ^* with coordinate q. The universal covering is the upper half plane $\mathbf{H}$ with coordinate t, and $q = e^{2\pi i t}$. Fix τ', and choose $\tau_0' \in \mathfrak{S}_g$. Consider the family $\pi : \mathcal{X}^*/\Delta^*$ defined by the holomorphic map $\tau(t) : \mathbf{H} \to \mathfrak{S}_g$

$$\tau_0' + \Im(\tau')t = (a\tau(t) + b\delta)(c\tau(t) + d\delta)^{-1}\delta.$$

This family has a multiplicative uniformization, which is a trivial algebraic torus $\widetilde{G}$ over Δ^*. For a point (x, y) in V, consider the image in $\widetilde{G}_t := V_t/X^* \cong (\mathbf{C}^*)^g$ for different t. We always use the dual basis of $\{v_1, \ldots, v_g\}$ as the standard coordinates of $(\mathbf{C}^*)^g$. The coordinates of (x, y) in $(\mathbf{C}^*)^g$ are given by the row vector

$$\exp\left(2\pi i(x\tau(t) + y\delta)(c\tau(t) + d\delta)^{-1}\right) = \exp\left(2\pi i(x\ y)\begin{pmatrix}\tau(t)\\ \delta\end{pmatrix}\Big(c\tau(t) + d\delta\Big)^{-1}\right).$$

Use the new coordinates (x', y'),

$$\exp\left(2\pi i\begin{pmatrix}x' & y'\end{pmatrix}\begin{pmatrix}A^{-1} & -A^{-1}B\\ 0 & I_g\end{pmatrix}\begin{pmatrix}(\tau_0' + \Im(\tau')t)\delta^{-1}\\ I_g\end{pmatrix}\right).$$

We only have to consider the periods $y' = 0$. Identify $Y = \Lambda/\Lambda \cap U$ with the subgroup $\langle u_1', \ldots, u_g'\rangle$ in Λ, and use the row vector (x') as the coordinates. It defines a bilinear map b from $Y \times X$ to holomorphic functions of t.

$$b(x', \alpha) = \exp\left(2\pi i x'(A^{-1}(\tau_0' + \Im(\tau')t)\delta^{-1} - A^{-1}B)\begin{pmatrix}\alpha(v_1)\\ \vdots\\ \alpha(v_g)\end{pmatrix}\right).$$

Here $\alpha \in X$, and x' means both the vector $\sum_{i=1}^{g} x'_i u'_i$ and its coordinates. Let's assume that the functions descend to Δ^*, and compute theirs orders of q. We use the argument principle. Notice that $\mathbf{H}$ is simply connected, and log is thus globally defined.

$$\begin{aligned}
&\operatorname{ord}_q(\exp(2\pi i x'(A^{-1}(\tau'_0 + \Im(\tau')t)\delta^{-1} - A^{-1}B))) \\
= \frac{1}{2\pi i}\int_\gamma &\,\mathrm{d}\log(\exp(2\pi i x'(A^{-1}(\tau'_0 + \Im(\tau')t)\delta^{-1} - A^{-1}B))) \\
&= \int_{[t_0, t_0+1]} \mathrm{d}(x'(A^{-1}(\tau'_0 + \Im(\tau')t)\delta^{-1} - A^{-1}B)) \\
&= x'A^{-1}\Im(\tau')\delta^{-1}.
\end{aligned}$$

Therefore, we have

Proposition 2.1. *The family descends to a holomorphic family over Δ^* if and only if*

$$A^{-1}\Im(\tau')\delta^{-1} \in \mathrm{M}(\mathbf{Z}).$$

Lemma 2.2.

$$x'A^{-1}\Im(\tau')\delta^{-1} = \Im((x\tau + y\delta)(c\tau + d\delta)^{-1}) = \check{\phi}(x, y).$$

Proof. Apply the transformation of coordinates,

$$\begin{aligned}
\Im((x\tau + y\delta)(c\tau + d\delta)^{-1}) &= \Im(\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix} \begin{pmatrix} \tau'\delta^{-1} \\ I_g \end{pmatrix}) \\
&= \Im(x'A^{-1}\tau'\delta^{-1}) \\
&= x'A^{-1}\Im(\tau')\delta^{-1}.
\end{aligned}$$

□

Corollary 2.3. The following statements are equivalent,

1) The tropical abelian variety $Tr(J)$ is integral,

2) $A^{-1}\Im(\tau')\delta^{-1} \in \mathrm{M}(\mathbf{Z})$,

3) the family defined above descends to a holomorphic family over Δ^*.

This justifies our Definition 3.24.

For simplicity, we also denote the coordinates of α by α. Decompose b

$$b(\lambda, \alpha) = b_0(\lambda, \alpha) \cdot b(t)(\lambda, \alpha) \cdot b'(\lambda, \alpha),$$

where

$$\begin{aligned} b_0(\lambda, \alpha) &= \exp\left(2\pi i x'(A^{-1}\tau_0'\delta^{-1})\alpha\right), \\ b_t(\lambda, \alpha) &= b(x', \alpha) = \exp\left(2\pi i x'(A^{-1}\Im(\tau')t\delta^{-1})\alpha\right) = q^{\langle \alpha, \check{\phi}(\lambda)\rangle}, \\ b'(\lambda, \alpha) &= \exp\left(-\pi i x' S \mathfrak{d}^{-1}\alpha\right). \end{aligned}$$

Fix the notation. $\phi(\lambda)(v) = E(\lambda, v)$. Therefore $\phi(u_i') = d_i v_i^*$. Notice that since $A^{-1}(\tau_0' + \Im(\tau')t)(A^{-1})^T$ is a symmetric matrix and S is integral skew symmetric, $b(\lambda_1, \phi(\lambda_2)) = b(\lambda_2, \phi(\lambda_1))$. Define the symmetric bilinear form $b_S : Y \times Y \to \mu_2 \cong \mathbf{Z}/2\mathbf{Z}$

$$b_S := \exp\left(-\pi i E(\lambda, \lambda')\right) = \exp\left(-\pi i x_1' S (x_2')^T\right) = \exp\left(\pi i x_1' S (x_2')^T\right) \equiv S \pmod 2.$$

Notice that $b_S(2\lambda_1, \lambda_2) = 1$ for any $\lambda_1, \lambda_2 \in Y$. Therefore b_S is a symmetric bilinear form over $Y/2Y$ which is a vector space over $\mathbf{Z}/2\mathbf{Z}$. There always exists some quadratic form χ whose associated bilinear form is b_S. For example we can define a symmetric matrix S' by requiring that $S'_{ij} = 1$ if S_{ij} is odd, and everywhere else is 0. Then

$$\chi(\lambda) := \exp\left(\frac{1}{2}\pi i x' S'(x')^T\right)$$

is such a quadratic form. There may be more than one choices of such a quadratic form (e.g. over characteristic 2). We fix a choice and denote it by χ. Let $\alpha' = \chi^{-1}$. We can even extend χ to get a semi-character $\chi : \Lambda \to \mathbb{S}^1$ by defining it to be trivial over $\Lambda \cap U$.

Define the positive quadratic forms Q_0 and Q by the symmetric matrices $1/2A^{-1}\tau_0'(A^{-1})^T$ and $1/2A^{-1}\Im(\tau')(A^{-1})^T$ respectively. Notice that

$$Q(\lambda) = \frac{1}{2}\langle \phi(\lambda), \check{\phi}(\lambda)\rangle.$$

Define

$$\begin{aligned} a(\lambda) &= \chi^{-1}(\lambda)\exp\left(2\pi i(Q_0(\lambda) + tQ(\lambda))\right) \\ &= a_0(\lambda) \cdot a_t(\lambda) \cdot a'(\lambda). \end{aligned}$$

where

$$\begin{aligned} a_0(\lambda) &:= \exp\left(2\pi i Q_0(\lambda)\right), \\ a_t(\lambda) &:= \exp\left(2\pi i t Q(\lambda)\right) = q^{Q(\lambda)}, \\ a'(\lambda) &:= \chi^{-1}(\lambda). \end{aligned}$$

Since Q is not necessarily integral, we may need to make a base change of degree 2 to make $a_t(\lambda)$ a function over Δ^*.

Lemma 2.4. *Use a, b above to define the Y-linearization of the trivial line bundle $\mathcal{O}_{\widetilde{G}}$ over $\widetilde{G}$, then $\mathcal{L}$, the descent of $\mathcal{O}_{\widetilde{G}}$ over $\mathcal{X}$, has the polarization of type δ.*

Proof. We do the computation explicitly. Use complex coordinates $z_\alpha = d_\alpha v_\alpha^*$. We can modify the 1-cocycle $\{e(\lambda, z)\}$ by a coboundary such that

$$e(u_i', z) = \exp(-2\pi i z_i).$$

Denote the matrix of $2Q_0 + 2Qt$ by $\tau''(t)$, and $\tau''(t) = X(t) + iY(t)$. Let $W(t) = Y(t)^{-1}$. We have

$$\begin{aligned} \mathrm{d}z_\alpha &= \sum_i (\tau''(t) - S/2)_{i\alpha}\, \mathrm{d}x_i' + d_\alpha\, \mathrm{d}y_\alpha' \\ \mathrm{d}\bar{z}_\beta &= \sum_j (\bar{\tau}''(t) - S/2)_{j\beta}\, \mathrm{d}x_j' + d_\beta\, \mathrm{d}y_\beta' \end{aligned}$$

Following ([GH94] pp. 310-311), we can define the hermitian metric by a transition function

$$h(z) = \exp\left(\frac{\pi}{2} \sum W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta - 2iY_{\beta\beta})\right)$$

Doing calculations similar to those in [GH94], we can check that h defines a hermitian metric for the 1-cocycle $\{e(\lambda, z)\}$. Moreover, the curvature of this hermitian metric is

$$\begin{aligned} \Theta_{\mathcal{L}} &= \pi \sum_{\alpha,\beta} W_{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \bar{z}_\beta \\ &= 2\pi i \Bigg(\sum_{\alpha>\beta} S_{\alpha\beta}\, \mathrm{d}x_\alpha' \wedge x_\beta' + \sum_\alpha d_\alpha\, \mathrm{d}x_\alpha' \wedge\, \mathrm{d}y_\alpha' \Bigg). \end{aligned}$$

This verifies that the degeneration data $\{a, b\}$ defines a line bundle $\mathcal{L}$ whose polarization is of type δ. $\square$

The limit Hodge filtration F_∞ is decided by $b_0 b'$. The log monodromy $N = \check{\phi} : V \to U$ is given by b_t. Although this family looks special, the general degenerating family with the same log monodromy N and limit Hodge filtration F_∞ is asymptotic to this family in a precise sense. This is the content of the nilpotent orbit theorem. Therefore we can use the data obtained from this family as a model for the general degeneration. Without loss of generality, we can choose τ'_0 to be 0, so that a_0 is 1. The conclusion is that a' is the twist necessary for the direction U, with respect to the basis $\{u'_i, v_j\}$.

2.2 General Degeneration

Consider a general one-parameter degeneration family $\mathcal{X}$ over Δ^*, whose abelian part is non-trivial. Let the associated isotropic subspace be $U_\xi \subset V$ of dimension $r \leqslant g$. U_ξ is obtained as the space of vanishing cycles $W_0^t \subset \Lambda_{\mathbf{R}}$. Let J be the complex structure on V. Let $\widetilde{T}$ be the subspace of V generated by U_ξ and $U_\xi J$. $T := \widetilde{T}/U_\xi \cap \Lambda$ is the toric part of dimension r. Since $U_\xi^\perp \cap U_\xi J = \{0\}$, $V/\widetilde{T} \cong U_\xi^\perp / U_\xi$ is a complex vector space of dimension $g' := g - r$. We have a pure Hodge structure of weight -1 on $U_\xi^\perp / U_\xi = W_1^t / W_0^t$. This is the period map of the abelian part A. The bilinear form E restricted to $U_\xi^\perp / U_\xi$ is non-degenerate. This gives the polarization on A. Let $\widetilde{G} = V/U_\xi^\perp \cap \Lambda$. $Y_\xi = \Lambda / U_\xi^\perp \cap \Lambda$. The family of abelian varieties $\mathcal{X}$ is the quotient of the family of semi-abelian varieties $\widetilde{G}$ by periods Y_ξ. We have the extension sequence of abelian group varieties over Δ^*

$$0 \longrightarrow T \longrightarrow \widetilde{G} \xrightarrow{\pi} A \longrightarrow 0. \tag{2.1}$$

Let X_ξ be the group of characters of T. It is the dual of the fundamental group $U_\xi \cap \Lambda$. For any $\alpha \in X_\xi$, α is a group homomorphism $\alpha : T \to \mathbb{G}_m$. The push-out of the short exact sequence (2.1) along α is a $\mathbb{G}_m$-torsor over A whose associated invertible sheaf is denoted by $\mathcal{O}_{-\alpha}$. Since the total space has a group structure, $\mathcal{O}_{-\alpha}$ is in $\mathrm{Pic}^0(A/\Delta^*)$. This defines the map $c : X_\xi \to A^t$, where A^t is the dual in the category of complex analytic spaces. Since Y_ξ is a group of Δ^*-sections of $\widetilde{G}$, for any $\lambda \in Y_\xi$, $\alpha \in X_\xi$, the push-out of λ along α is a Δ^*-section of $\mathcal{O}_{-\alpha}$. Denote the projection under π by $c^t(\lambda) \subset A$, and the section by $\tau(\lambda, \alpha)$. This gives the trivialization τ of the biextension $(c^t \times c)^* \mathcal{P}^{-1}$.

The relatively ample line bundle of type δ on $\mathcal{X}$ is represented as a line bundle $\widetilde{\mathcal{L}}$ over $\widetilde{G}$ with a Y_ξ-action. Since $\widetilde{G}$ is a T-torsor, $\widetilde{\mathcal{L}}$ is also equipped with a T-action, and we can do a partial Fourier expansion. Restricted to any section of A, $\widetilde{\mathcal{L}}$ is trivial. Therefore, suppose that $\widetilde{\mathcal{L}}$ descends to an ample line bundle $\mathcal{M}$ of A. This is a choice, and we fix this choice. Then the α-eigenspace of $\Gamma(\widetilde{G}, \widetilde{\mathcal{L}})$ is identified with $\Gamma(A, \mathcal{M}_\alpha)$ and the partial Fourier expansion is

$$\Gamma(\widetilde{G}, \widetilde{\mathcal{L}}) = \bigoplus_{\alpha \in X'} \Gamma(A, \mathcal{M}_\alpha).$$

Denote the Fourier coefficients by the homomorphisms $\sigma_\alpha : \Gamma(\widetilde{G}, \widetilde{\mathcal{L}}) \to \Gamma(A, \mathcal{M}_\alpha)$. Restricting to the Y_ξ-invariant subspace $\Gamma(\mathcal{X}, \mathcal{L})$, there is a relation between the homomorphisms σ_α and $\sigma_{\alpha+\phi(\lambda)}$ for $\lambda \in Y_\xi$. That is,

$$\sigma_{\alpha+\phi(\lambda)} = \psi(\lambda)\tau(\lambda, \alpha)T^*_{c^t(\lambda)} \circ \sigma_\alpha \tag{2.2}$$

for $\psi(y)$ a Δ^*-section of $\mathcal{M}(c^t(\lambda))^{-1}$. This defines the trivialization ψ of the central extension over Y_ξ.

To get the explicit data τ and ψ, we choose a 0-cusp $F(U)$ that is in the closure of $F(U_\xi)$, i.e. a maximal rational isotropic subspace U that contains U_ξ. Assume $\Lambda \cap U_\xi$ be spanned by $\{v_1, v_2, \ldots, v_r\}$, and $U \cap \Lambda$ be spanned by $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_g\}$. We choose the complement $\{u'_1, \ldots, u'_g\}$ as in the above section. Therefore $U_\xi^\perp \cap \Lambda$ is spanned by

$$\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_g, u'_{r+1}, \ldots, u'_g\}.$$

For simplicity, assume the periods for the family are given by $\tau'_0 + t\Im(\tau')$ after the transformation by M. Write τ'_0 in blocks.

$$\tau'_0 = \begin{pmatrix} \tau'_1 & \tau'_2 \\ \tau'_3 & \tau'_4 \end{pmatrix},$$

where τ'_1 is a $r \times r$-matrix and τ'_4 is a positive-definite $g' \times g'$-matrix. Similarly, we write S in blocks. Assume $\Im(\tau')$ is positive definite for the upper left $r \times r$ block and vanishes anywhere else. As in the above model, we take the quotient of the group generated by $\{v_1, \ldots, v_g\}$ and get an algebraic torus $\mathbb{G}_m^g$

over Δ^*. Denote the group generated by $\{u'_{r+1}, \ldots, u'_g\}$ by Y' and the group generated by $\{u'_1, \ldots, u'_r\}$ by Y_ξ. Use λ for a vector in Y_ξ and z for a vector in Y'. Use X^α for coordinate functions of $\mathbb{G}_m^r$, and W^β for coordinate functions on $\mathbb{G}_m^{g'}$. The family of abelian varieties A is the quotient of $\mathbb{G}_m^{g'}$ by Y'. $\widetilde{G}$ is the quotient of $\mathbb{G}_m^g$ by Y'. By this multiplicative uniformization $\mathbb{G}_m^g$, all line bundles $\mathcal{O}_\alpha$ are trivialized canonically after pull back over $\mathbb{G}_m^{g'}$. Of course there is ambiguity from the action of Y' for the trivialization of every fiber $\mathcal{O}_\alpha(c^t(\lambda))$. However, the lift of Y_ξ to points $u'_1, \ldots, u'_g$ fixes this ambiguity. The upshot is $\mathcal{O}_\alpha^{-1}(c^t(\lambda))$ is thus trivialized, and the section $\tau(\lambda, \alpha)$ is equivalent to a function over Δ^*. Similarly, the pull back of $\mathcal{M}$ to $\mathbb{G}_m^{g'}$ is trivial. A lift of Y_ξ gives a canonical trivialization of $\mathcal{M}^{-1}(c^t(\lambda))$. The section $\psi(\lambda)$ should also be a function over Δ^*.

Regard $\widetilde{\mathcal{L}}$ as the quotient of $(\mathbf{C}^*)^r \times (\mathbf{C}^*)^{g'} \times \mathbf{C} \times \Delta^*$ by Y'. The action is parametrized by $q \in \Delta^*$. A section $\vartheta \in \Gamma(\mathcal{X}, \mathcal{L})$ is a Y-invariant function over $(\mathbf{C}^*)^r \times (\mathbf{C}^*)^{g'}$. Do the partial Fourier decomposition

$$\vartheta = \sum_{\alpha \in X_\xi} \left(\sum_\beta a_{\alpha\beta} \mathrm{W}^\beta\right) \mathrm{X}^\alpha. \tag{2.3}$$

The function over $(\mathbf{C}^*)^{g'}$

$$\sigma_\alpha(\vartheta) = \sum_\beta a_{\alpha\beta} \mathrm{W}^\beta$$

is a section of $\Gamma(A, \mathcal{M}_\alpha)$.

This can be easily verified. We use e, a, b from the above section. For $\mu \in Y$, we have

$$\vartheta((\mathrm{W},\mathrm{X})+\mu) = e(\mu,(\mathrm{W},\mathrm{X}))\vartheta(\mathrm{W},\mathrm{X}) = \frac{1}{a(\mu)b((\mathrm{W},\mathrm{X}),\phi(\mu))}\vartheta(\mathrm{W},\mathrm{X}).$$

If $\mu = z \in Y'$,

$$b((\mathrm{W},\mathrm{X}),\phi(z)) = \mathrm{W}^{\phi(z)}.$$

On the one hand,

$$\vartheta((\mathrm{W},\mathrm{X})+z) = \sum_{\alpha\in X_\xi}(\sum_{\beta} a_{\alpha\beta}b(z,\beta)\mathrm{W}^{\beta})b(z,\alpha)\mathrm{X}^{\alpha} \tag{2.4}$$

$$= \sum_{\alpha\in X_\xi}\sigma_\alpha(\vartheta)(\mathrm{W}+z)b(z,\alpha)\mathrm{X}^{\alpha}. \tag{2.5}$$

On the other hand,

$$e(z,(\mathrm{W},\mathrm{X}))\vartheta(\mathrm{W},\mathrm{X}) = \frac{1}{a(z)\mathrm{W}^{\phi(z)}}\sum_{\alpha\in X_\xi}\sigma_\alpha(\vartheta)(\mathrm{W})\mathrm{X}^{\alpha}. \tag{2.6}$$

Compare the coefficients of X^{α},

$$\sigma_\alpha(\vartheta)(\mathrm{W}+z) = \frac{1}{b(z,\alpha)a(z)\mathrm{W}^{\phi(z)}}\sigma_\alpha(\vartheta)(\mathrm{W}) \tag{2.7}$$

Here $1/b(z,\alpha)$ are the factors of automorphy for $\mathcal{O}_\alpha$, and $1/(a(z)\mathrm{W}^{\phi(z)})$ are the factors of automorphy for $\mathcal{M}$. Hence $\sigma_\alpha(\vartheta)$ is a section of $\mathcal{M}_\alpha$. The abelian part A is determined by $b(z,\beta)$, i.e. the periods τ_4' and the twist S_4. We can also see the twist for $\mathcal{O}_\alpha$ in Equation (4.23) is

$$\exp(-\pi i z S_3 \mathfrak{d}_1^{-1}\alpha).$$

If $\mu = \lambda \in Y_\xi$,

$$b(\mathrm{W},\mathrm{X},\phi(\lambda)) = \mathrm{X}^{\phi(\lambda)}.$$

We have

$$\vartheta((\mathrm{W},\mathrm{X})+\lambda) = \sum_{\alpha\in X_\xi}\Big(\sum_{\beta} a_{\alpha\beta} b(\lambda,\beta)\mathrm{W}^\beta\Big) b(\lambda,\alpha)\mathrm{X}^\alpha \tag{2.8}$$

$$= \sum_{\alpha\in X_\xi}\sigma_\alpha(\vartheta)(\mathrm{W}+c^t(\lambda))b(\lambda,\alpha)\mathrm{X}^\alpha. \tag{2.9}$$

And

$$e(\lambda,(\mathrm{W},\mathrm{X}))\vartheta(\mathrm{W},\mathrm{X}) = \frac{1}{a(\lambda)}\sum_{\alpha\in X_\xi}\sigma_\alpha(\vartheta)(\mathrm{W})\mathrm{X}^{\alpha-\phi(\lambda)}. \tag{2.10}$$

Compare the coefficients of X^α,

$$\sigma_{\alpha+\phi(\lambda)}(\vartheta)(\mathrm{W}) = a(\lambda)b(\lambda,\alpha)\sigma_\alpha(\vartheta)(\mathrm{W}+c^t(\lambda)), \tag{2.11}$$

$$= a(\lambda)b(\lambda,\alpha)T^*_{c^t(\lambda)}\circ\sigma_\alpha(\vartheta)(\mathrm{W}). \tag{2.12}$$

Therefore, after using the canonical trivialization from the 0-cusp U,

$$\psi(\lambda) = a(\lambda) \tag{2.13}$$

$$\tau(\lambda, \alpha) = b(\lambda, \alpha). \tag{2.14}$$

Using the coordinates with respect to $\{u_1', \ldots, u_r'\}$, the points are

$$\begin{aligned} b(x, \alpha) &= \exp\left(2\pi i(x, 0)(A^{-1}(\tau_0' + \Im(\tau')t)\delta^{-1} - 1/2S\mathfrak{d}^{-1})\begin{pmatrix} v_1(\alpha) \\ \vdots \\ v_r(\alpha) \\ 0 \end{pmatrix}\right) \\ &= b_0 b'(x, \alpha) b_t(x, \alpha), \end{aligned}$$

where $b_0 b'(x, \alpha)$ is a constant in $\mathbf{C}$, while $b_t(x, \alpha)$ is a function over Δ^*. Write b_t as an $r \times r$-matrix of functions

$$q^{2Q\mathfrak{d}^{-1}}.$$

Write

$$a(x) = a_0 a'(x) a_t(x),$$

where $a_0 a'(x)$ is a constant, and $a_t(x) = q^{Q(x)}$ is a function of q if we make a necessary base change.

Since the quadratic form $Q \in \mathcal{C}(X_\xi)$,

Corollary 2.5. The trivializations τ and ψ are compatible with Q.

Since we will need to do approximation anyway, assume $a_0 = b_0 = 1$. Denote the function field on Δ^* by K. The explicit data for U_ξ is

Proposition 2.6. *If we make the choice of the* 0*-cusp* U*, and use the data from* U*, we can write* τ*,* ψ *as follows.*

$$\tau = b' b_t : Y_\xi \times X_\xi \to K \tag{2.15}$$

$$\psi = a' a_t : Y_\xi \to K, \tag{2.16}$$

where $b_t = q^{2Q\mathfrak{d}^{-1}}$ *and* $a_t = q^{Q(x)}$ *is defined in terms of* $Q \in \mathcal{C}(X_\xi)$*, and* b'*,* a' *are twists associated to the boundary component* U_ξ

$$b'(\lambda, \alpha) = \exp\left(- \pi i x(\lambda) S_1 \mathfrak{d}^{-1} \begin{pmatrix} v_1(\alpha) \\ \vdots \\ v_r(\alpha) \end{pmatrix} \right), \tag{2.17}$$

$$a'(\lambda) = \exp\left(-1/2 \pi i x(\lambda) S_1' x(\lambda)^T\right), \tag{2.18}$$

where S_1' *is a symmetric matrix such that* $S_1' \equiv S_1 \pmod{2\mathbf{Z}}$.

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